# Extending Qualitative Reconstruction Methods to Biharmonic Wave Scattering

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## Introduction

We study the inverse shape problem of reconstructing unknown clamped cavities from flexural wave fields governed by the biharmonic wave operator. We adapt two established techniques to this setting: the **Linear Sampling Method (LSM)**, which uses full multistatic data to probe the interior of scatterers, and the **Extended Sampling Method (ESM)**, a recently developed approach designed for limited-aperture data.

## Problem Setup

We study flexural wave scattering from a clamped cavity $D \subset \mathbb{R}^2$ with smooth boundary $\partial D$.

**Governing PDE:**

$$\Delta^2 u^s - k^4 u^s = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{D}$$

**Boundary conditions (clamped):**

$$u^s = -u^i, \quad \partial_\nu u^s = -\partial_\nu u^i \quad \text{on } \partial D,$$

where the incident field is the plane wave $u^i(x) = e^{ikx\cdot d}$.

**Radiation condition:**

$$\lim_{r\to\infty} \sqrt{r}\,(\partial_r u^s - iku^s) = 0, \quad r = |x|.$$

Decompose the scattered field:

$$u^s = u^s_H + u^s_M, \quad \begin{cases} \Delta u^s_H + k^2 u^s_H = 0, \\ \Delta u^s_M - k^2 u^s_M = 0, \end{cases} \quad \mathbb{R}^2 \setminus \overline{D}.$$

Coupled boundary conditions:

$$u^s_H + u^s_M = -u^i, \quad \partial_\nu u^s_H + \partial_\nu u^s_M = -\partial_\nu u^i \text{ on } \partial D.$$

**Asymptotic behavior:**

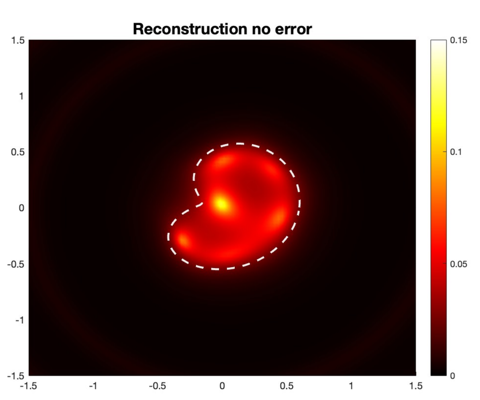
$$|u^s_H| = \mathcal{O}(r^{-1/2}), \quad |u^s_M| = \mathcal{O}(e^{-kr} r^{-1/2}).$$

The far-field pattern $u^\infty(\hat{x}, d)$ satisfies:

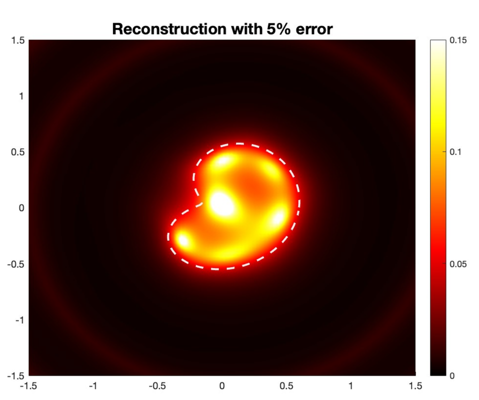
$$u^s(x) = \frac{e^{ik|x|}}{\sqrt{|x|}} u^\infty(\hat{x}, d) + \mathcal{O}(|x|^{-3/2}).$$

**Inverse problem:** Recover $D$ from $u^\infty(\hat{x}, d) = u^\infty_H(\hat{x}, d)$ for all $\hat{x}, d \in \mathbb{S}^1$.

## Select LSM Reconstructions

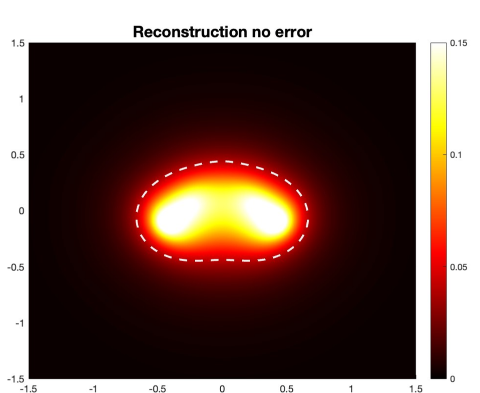



**Apple-shaped cavity,** $\kappa = 2\pi$, no noise, 30 incident/observation directions, $250 \times 250$ grid.

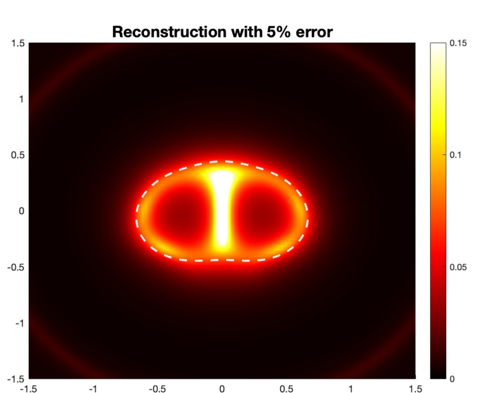



**Apple-shaped cavity,** $\kappa = 2\pi$, noise $\delta = 0.05$, 30 incident/observation directions, $250 \times 250$ grid.

**Parametrization of Apple:**
$\gamma(t) = \frac{0.55(1+0.9\cos t + 0.1 \sin 2t)}{1+0.75\cos t}(\cos t, \sin t)$




**Peach-shaped cavity,** $\kappa = \pi$, no noise, 30 incident/observation directions, $250 \times 250$ grid.




**Peach-shaped cavity,** $\kappa = 2\pi$, noise $\delta = 0.05$, 30 incident/observation directions, $250 \times 250$ grid.

**Parametrization of Peach:**
$\gamma(t) = 0.22(\cos^2 t\sqrt{1-\sin t} + 2)(\cos t, \sin t)$

## Linear Sampling Method (LSM)

**Goal:** Determine whether a sampling point $z \in \mathbb{R}^2$ lies inside the unknown cavity $D \subset \mathbb{R}^2$.

**LSM Equation:**

$$\mathcal{F} g_z = G^\infty_\kappa(\cdot, z), \quad (\mathcal{F} g_z)(\hat{x}) := \int_{\mathbb{S}^1} u^\infty(\hat{x}, d)\, g_z(d)\, ds(d)$$

**Operator Factorization:**

$$\mathcal{F} = \mathcal{G} \circ (-\mathcal{H}), \quad z \in D \iff e^{i\kappa\hat{x}\cdot z} \in \text{Range}(\mathcal{G})$$

Where:

$$\mathcal{H} g_z := (v_{g_z}, \partial_\nu v_{g_z})^\top, \; v_g(x) := \int_{\mathbb{S}^1} e^{i\kappa x\cdot d} g(d)\, ds(d), \quad \text{and } \mathcal{G}(h_1, h_2) = w^\infty,$$

with $w \in H^2_{\text{loc}}(\mathbb{R}^2 \setminus \overline{D})$ the unique weak solution to the generalized biharmonic scattering problem

$$\begin{cases} (\Delta^2 - \kappa^4) w = 0 & \text{in } \mathbb{R}^2 \setminus \overline{D}, \\ (w, \partial_\nu w) = (h_1, h_2) & \text{on } \partial D, \\ \lim\limits_{r\to\infty} \sqrt{r}(\partial_r w - i\kappa w) = 0, & r = |x|. \end{cases}$$

- $\mathcal{F}$: Far-field operator mapping weights $g_z$ to superpositions of measured data.
- $G^\infty_\kappa(\cdot, z)$: Far-field pattern of a biharmonic point source at $z$.

**Sampling Principle:**

- *Feasible* (regularized) solutions $g_z$ exist with small norm if and only if $z \in D$.
- *Indicator:* Plotting $\|g_z\|_{L^2}$ reveals the support of $D$.

**Theory Insight: Approximate Solvability**

- Far-field operator $\mathcal{F}$: $L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is compact.
- Compactness from analyticity of $u^\infty(\hat{x}, d)$.
- Approximate solutions exist if $\mathcal{F}$ is injective with dense range.
- This holds if a related clamped transmission problem has only the trivial solution.

*This justifies use of norm of $g_z$ as an indicator.*

**Clamped Transmission Problem:** Find

$$(p, q) \in H^1(\mathbb{R}^2 \setminus \overline{D}) \times H^1(D) \text{ s.t.}$$

$$\begin{cases} (\Delta - \kappa^2) p = 0 & \text{in } \mathbb{R}^2 \setminus \overline{D} \\ (\Delta + \kappa^2) q = 0 & \text{in } D \\ q + p = 0, \quad \partial_\nu q + \partial_\nu p = 0 & \text{on } \partial D \end{cases}$$

**Reciprocity:** $u^\infty(\hat{x}, d) = u^\infty(-d, -\hat{x})$
$\Rightarrow$ *$\mathcal{F}$ injective implies $\mathcal{F}^*$ injective.*

**Conclusion:** If $\kappa^2$ is not a clamped eigenvalue, then $\mathcal{F}$ is injective with dense range in $L^2(\mathbb{S}^1)$.

## Extended Sampling Method (ESM)

**Goal:** Approximate the location of the cavity $D \subset \mathbb{R}^2$ using only *one* incident plane wave or limited aperture data.

**Idea:** Compare measured far-field data with known test fields from artificial obstacles centered at sampling points $z \in \mathbb{R}^2$.

**Test Field:** Let $U^\infty_{B_z}(\hat{x}, \hat{y})$ be the far-field pattern of a sound-soft disk centered at $z$. It is given by:

$$U^\infty_{B_z}(\hat{x}, \hat{y}) = e^{i\kappa z\cdot(\hat{y}-\hat{x})} U^\infty_{B_0}(\hat{x}, \hat{y})$$

**Known Kernel:** The unshifted field $U^\infty_{B_0}$ admits a closed-form expansion involving Bessel and Hankel functions.

**ESM Equation:**

$$\mathcal{F}_{B_z} g_z = u^\infty, \quad (\mathcal{F}_{B_z} g_z)(\hat{x}) = \int_{\mathbb{S}^1} U^\infty_{B_z}(\hat{x}, \hat{y})\, g_z(\hat{y})\, ds(\hat{y})$$

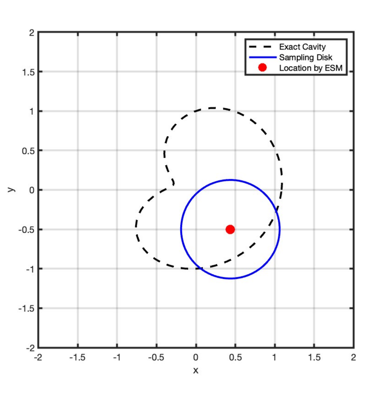
**Indicator Principle:**

- *Small-norm* solutions $g_z$ exist only if $D \subseteq B_z$.
- No need for full multistatic data—only one incident direction suffices.

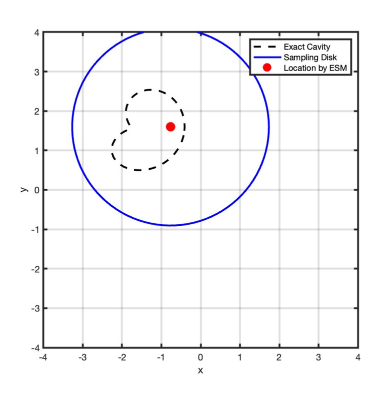
## Select Multilevel ESM Results

*Multilevel strategy improves accuracy by zooming in on the cavity through iterative radius refinement. Fixed direction $d = (1/2, \sqrt{3}/2)$.*

**Apple-shaped cavity location:**



Apple at origin



Apple at $(-1.5, 1.5)$

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