# MA 16020 – Applied Calculus II: Lecture 9
# Integration by Partial Fractions

# Partial Fractions: Motivation

So far we have learned:

- U-substitution (*reverse chain rule*)
- Integration by parts (*reverse product rule*)

Now we will learn **Partial Fractions**: a method for *integrating rational functions*.

A rational function is of the form:

$$f(x) = \frac{P(x)}{Q(x)}, \quad \text{where } P \text{ and } Q \text{ are polynomials.}$$

**Recall: Adding fractions** with different denominators:

$$\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}.$$

This idea motivates the *partial fraction decomposition*: we write a complicated fraction as a sum of simpler fractions.

We can extend the idea to functions of $x$.

**Example 1:** Combine the fractions

$$\frac{1}{x-2} + \frac{3}{x-5}.$$

**Step 1: Find the least common denominator (LCD):**

$$\text{LCD} = (x-2)(x-5)$$

**Step 2: Combine:**

$$\frac{1}{x-2} + \frac{3}{x-5} = \frac{(x-5)+3(x-2)}{(x-2)(x-5)} = \frac{4x-11}{(x-2)(x-5)}.$$

This shows how a single fraction can be *decomposed* into simpler fractions — the reverse of adding fractions.

# Why U-Substitution Fails

Consider the integral:

$$\int \frac{4x - 11}{(x-2)(x-5)}\, dx$$

**Attempt:** Let $u = (x-2)(x-5)$, then

$$du = (2x - 7)dx.$$

**Problem:** The numerator $4x - 11$ *does not match* $du = 2x - 7$.

Conclusion: U-substitution alone fails — we need **partial fraction decomposition** to break this into integrable pieces.

# Partial Fractions: Integration Example

Recall from the previous example, after combining fractions we have:

$$\frac{4x-11}{(x-2)(x-5)} = \frac{1}{x-2} + \frac{3}{x-5}.$$

**Step 1: Integrate term by term**

$$\int \frac{4x-11}{(x-2)(x-5)}\,dx = \int \frac{1}{x-2}\,dx + \int \frac{3}{x-5}\,dx$$

**Step 2: Apply basic logarithm rule**

$$\int \frac{1}{x-a}\,dx = \ln|x-a| + C$$

**Step 3: Write the final result**

$$\int \frac{4x-11}{(x-2)(x-5)}\,dx = \ln|x-2| + 3\ln|x-5| + C$$

Observation: Partial fraction decomposition makes the integration straightforward.

# Partial Fractions: Four Types

**Rational functions:** $\frac{P(x)}{Q(x)}$, where $P$ and $Q$ are polynomials.

**There are four main types of partial fractions:**

1. Distinct linear factors: $(x-a)(x-b)\cdots$
2. Repeated linear factors: $(x-a)^n$
3. Distinct irreducible quadratic factors: $(x^2+bx+c)$
4. Repeated irreducible quadratic factors: $(x^2+bx+c)^n$

Today we will focus on Type 1 and Type 2.

**Type 1: Distinct Linear Factors Rule** If the denominator factors into distinct linear terms:

$$\frac{P(x)}{(x-a)(x-b)\cdots} = \frac{A}{x-a} + \frac{B}{x-b} + \cdots$$

Constants $A, B, \ldots$ are solved by clearing denominators and either:

- Plugging in convenient values of $x$, or
- Matching coefficients.

**Example 2:** Find the partial fraction decomposition of the following Type 1 rational functions:

1. $\dfrac{3x+5}{(x-1)(x+2)}$
2. $\dfrac{2x+7}{(x+1)(x-3)}$
3. $\dfrac{5x+10}{(x-2)(x+3)}$

Note: Solve for the constants $A, B$ by clearing denominators and either plugging in convenient values of $x$ or matching coefficients.

# Type 1 Partial Fraction Integration

**Example 3:** Use the results of **Example 2** to evaluate the integrals of the following Type 1 rational functions:

1. $\displaystyle\int \frac{3x+5}{(x-1)(x+2)}\,dx$
2. $\displaystyle\int \frac{2x+7}{(x+1)(x-3)}\,dx$
3. $\displaystyle\int \frac{5x+10}{(x-2)(x+3)}\,dx$

**Type 2: Repeated Linear Factors**

When the denominator contains powers of the same linear factor, e.g. $(x-a)^n$, the partial fraction decomposition requires a term for *each power*:

$$\frac{P(x)}{(x-a)^n} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_n}{(x-a)^n}.$$

**Key points:**

- Each $A_i$ is a constant to solve.
- Multiply through by the denominator and either:
    - Plug in convenient values of $x$ (especially $x = a$ for the highest power term), or
    - Match coefficients.

Next time we will do simple examples to illustrate.

# Type 2 Partial Fraction Examples

Find the partial fraction decomposition of the following Type 2 rational functions and evaluate the integrals:

1. $\dfrac{3x+5}{x^2-6x+9}$ (Hint: factor denominator if possible)
2. $\dfrac{3x+7}{(x-2)^2}$

**Steps:**

- Factor the denominator (if repeated linear factor, write separate terms for each power)
- Set up partial fraction decomposition
- Solve for constants
- Integrate each term

(Constants and integration to be done on board)