MA 16020 – Applied Calculus II: Lecture 6
Integration by Substitution (Logarithms)

Substitution and the Log Function

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- More generally:

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- This is the **u-substitution rule for logarithms**. Let $u = g(x)$, then $du = g'(x)\,dx$.

Example 1

Compute:

$$\int \frac{2x}{x^2+1}\,dx$$

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- Substitution: $u = x^2 + 1 \quad \Rightarrow \quad du = 2x\,dx.$
- Integral becomes:

$$\int \frac{du}{u} = \ln|u| + C = \ln(x^2+1) + C.$$

Example 2

Compute:

$$\int \frac{5}{3x-7}\,dx$$

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- Substitution: $u = 3x - 7 \quad \Rightarrow \quad du = 3\,dx.$
- Factor constants:

$$\int \frac{5}{3x-7}\,dx = \frac{5}{3}\int \frac{du}{u}.$$

- Answer:

$$\frac{5}{3}\ln|3x-7| + C.$$

General Strategy

1. Look for a fraction of the form $\dfrac{g'(x)}{g(x)}$.
2. If found, set $u = g(x)$.
3. Rewrite the integral in terms of u.
4. Integrate: $\int \frac{1}{u}\,du = \ln|u| + C$.
5. Substitute back $u = g(x)$.

Practice Problems

Try the following:

(a) $$\int \frac{6x}{x^2 - 4}\, dx$$

(b) $$\int \frac{1}{5x + 2}\, dx$$

(c) $$\int \frac{3x^2}{x^3 + 7}\, dx$$

Trig Examples: Logarithmic Forms

Recall: the same rule applies to trig functions.

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Examples:

1. $\displaystyle\int 13\cot\frac{x}{5}\,dx$
2. $\displaystyle\int \frac{\cos x}{\sin x}\,dx$
3. $\displaystyle\int \frac{\sec^2 x}{\tan x}\,dx$
4. $\displaystyle\int \frac{\csc^2 x}{\cot x}\,dx$

Example: $\int \frac{\cos x}{\sin x}\, dx$

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- Answer:

$$\ln|\sin x| + C.$$

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- Substitution: $u = \tan x \quad \Rightarrow \quad du = \sec^2 x\, dx.$
- Integral becomes:

$$\int \frac{\sec^2 x}{\tan x}\, dx = \int \frac{du}{u}.$$

- Answer:

$$\ln|\tan x| + C.$$

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- Substitution: $u = \cot x \quad \Rightarrow \quad du = -\csc^2 x\, dx$.
- Integral becomes:

$$\int \frac{\csc^2 x}{\cot x}\, dx = -\int \frac{du}{u}.$$

- Answer:

$$-\ln|\cot x| + C = \ln|\tan x| + C.$$

Additional Practice

Compute the following indefinite integrals.

- $\displaystyle\int \frac{dx}{5x(\ln 3x)^2}$
- $\displaystyle\int \frac{dx}{x(\ln x^{14})}$
- $\displaystyle\int \frac{4(\ln x)^3}{x}\,dx$
- $\displaystyle\int \frac{\ln\sqrt{x}}{x}\,dx$

Solution 1

Compute $\displaystyle\int \frac{dx}{5x(\ln 3x)^2}$.

Solution: Let $u = \ln 3x \implies du = \frac{1}{x}dx$.

$$\int \frac{dx}{5x(\ln 3x)^2} = \frac{1}{5}\int \frac{du}{u^2} = \frac{1}{5}\left(-\frac{1}{u}\right) + C = -\frac{1}{5\ln 3x} + C$$

Solution 2

Compute $\displaystyle\int \frac{dx}{x(\ln x^{14})}$.

Solution: Notice $\ln x^{14} = 14 \ln x$. Let $u = \ln x \implies du = \frac{dx}{x}$.

$$\int \frac{dx}{x(\ln x^{14})} = \int \frac{dx}{x(14 \ln x)} = \frac{1}{14} \int \frac{du}{u} = \frac{1}{14} \ln |u| + C = \frac{1}{14} \ln |\ln x| + C$$

Solution 3

Compute $\displaystyle\int \frac{4(\ln x)^3}{x} dx$.

Solution: Let $u = \ln x \implies du = \frac{dx}{x}$.

$$\int \frac{4(\ln x)^3}{x} dx = 4 \int u^3 du = 4 \cdot \frac{u^4}{4} + C = (\ln x)^4 + C$$

Solution 4

Compute $\displaystyle\int \frac{\ln\sqrt{x}}{x}\,dx$.

Solution: Note $\ln\sqrt{x} = \frac{1}{2}\ln x$. Let $u = \ln x \implies du = \frac{dx}{x}$.

$$\int \frac{\ln\sqrt{x}}{x}\,dx = \int \frac{1}{2}u\,du = \frac{1}{4}u^2 + C = \frac{1}{4}(\ln x)^2 + C$$