MA 16020 – Applied Calculus II: Lectures 26-27, Functions of Two Variables & Intro to Partial Derivatives

A **function of two variables** assigns a number to each pair (x, y): we write

$$z = f(x, y)$$

- x and y are independent variables.
- $f(x, y)$ is the dependent variable (the output).
- Input: A point (x, y)
- Output: A number $f(x, y) = z$

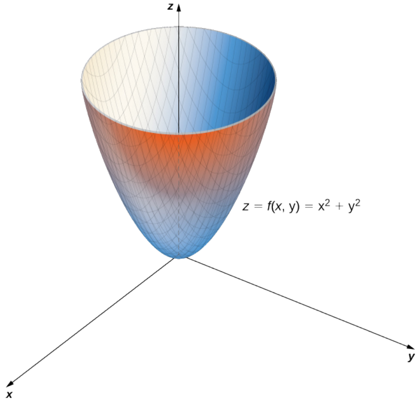
Example:

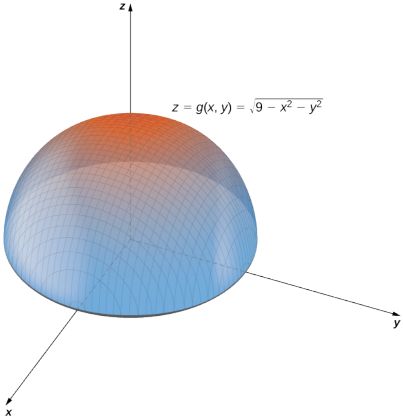
$$z = f(x, y) = x + y$$

- If $x = 2$ and $y = 3$, then $z = f(2, 3) = 5$.

Visualizing Functions of Two Variables

- The graph of $f(x, y)$ is a surface in 3D.





Surface 1: $z = f(x, y) = x^2 + y^2$

Surface 2:
$z = g(x, y) = \sqrt{9 - x^2 - y^2}$

Example 1. Compute the indicated functional value.

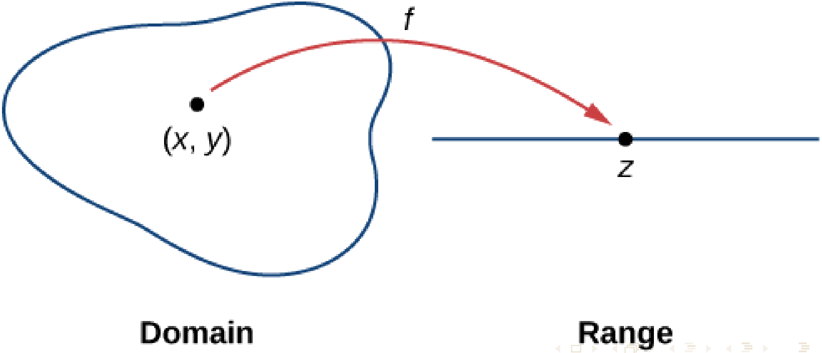
1. $$f(x,y) = \frac{3x+2y}{2x+3y}; \quad f(-4,6)$$

2. $$f(x,y) = \frac{e^{xy}}{2-e^{xy}}; \quad f(1,\ln 1),\ f(\ln 2, 2)$$

Domain & Range

Just as for functions of a single variable, we can find the domain and range of functions of two variables in a similar fashion.
The main difference for two variables is

1. **Domain:** All points (x, y) in the xy-plane for which the function $f(x, y)$ is well-defined. The domain will be a subset of the plane.
2. All values that the function $z = f(x, y)$ produces.



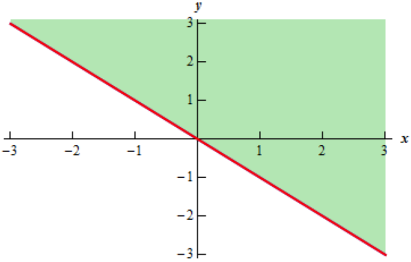
Example 2. Find the domain and range of the following functions. Sketch a graph of the domain.

1. $$f(x, y) = \sqrt{x + y}$$
2. $$f(x, y) = \sqrt{x} + \sqrt{y}$$
3. $$f(x, y) = \ln\left(9 - x^2 - 9y^2\right)$$

We know the square root of a negative number is undefined. So

$$f(x, y) = \sqrt{x + y} \text{ defined if } x + y \geq 0$$

So Domain: $\{(x, y) : x + y \geq 0, x, y \in \mathbb{R}\}$.



Range: $\{z = f(x, y) : z \geq 0\}$

Level Curves of a Function

Definition: Level Curves

For a function of two variables $f(x, y)$, a **level curve** (or contour line) is the set of points (x, y) in the domain where the function takes a constant value c:

$$f(x, y) = c$$

Each value of c gives a different level curve.

Example: For $f(x, y) = x^2 + y^2$, the level curves are

$$x^2 + y^2 = c$$

which are circles of radius $\sqrt{c}$ centered at the origin.

Example 3: Sketch Level Curves

Consider the function

$$f(x, y) = \sqrt{x^2 + y^2}.$$

- Sketch the level curves corresponding to

  $$c = 0, 1, 2, 3$$

  that is, the sets of points (x, y) where

  $$f(x, y) = c.$$

- Use different colors for each value of c to clearly distinguish the curves.

Hint: These level curves are circles centered at the origin with radius c.

E

xample 4. What do the level curves for $f(x, y) = 11\sqrt{y + 6x^2}$ look like?

- Key. Let $f(x, y) = c$. Let's solve for y.

Example 5. What does the level curve for $f(x, y) = \ln(x^2 + y^2)$ at $c = \ln 36$ look like?

The Partial Derivative for a Function of Two Variables

Consider a Surface $z = f(x, y)$

We can move in the x-direction or the y-direction. To measure the slope along one direction, we **hold the other variable constant**.

Example

Let $f(x, y) = x^2 + y^2$.

$$f_x(x, y) = 2x \qquad \text{(hold } y \text{ constant)}$$
$$f_y(x, y) = 2y \qquad \text{(hold } x \text{ constant)}$$

To understand what it means to treat y as a constant, pick a value.

$$f(x, 5) = x^2 + 5^2.$$

Then

$$f_x(x, 5) = 2x + 0 = 2x.$$

Partial Derivative = Derivative with One Variable Fixed

Example

Let $f(x, y) = 3x^2 + 2y^2$.

$$f_x(x, y) = \frac{\partial}{\partial x}(3x^2 + 2y^2) = 6x,$$

$$f_y(x, y) = \frac{\partial}{\partial y}(3x^2 + 2y^2) = 4y.$$

(The blue term is treated as a constant.)

At a Point

At $(x, y) = (1, 2)$:

$$f_x(1, 2) = 6, \qquad f_y(1, 2) = 8.$$

So the surface rises faster in the y-direction at that point.

Partial Derivatives as Limits

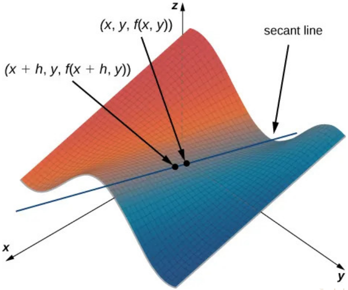
Definition

For a function $f(x, y)$, the **partial derivatives** are defined by

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h},$$

and

$$f_y(x, y) = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}.$$



Notations for Partial Derivative

Partial derivative of f with respect to x

The following notations all mean the same thing:

$$f_x = \frac{\partial f}{\partial x} = \partial_x f = z_x = \frac{\partial z}{\partial x} = D_x f$$

Partial derivative of f with respect to y

The following notations all mean the same thing:

$$f_y = \frac{\partial f}{\partial y} = \partial_y f = z_y = \frac{\partial z}{\partial y} = D_y f$$

Example 6: Computing Partial Derivatives

Compute the partial derivatives with respect to x and y.

1. $$z = f(x, y) = x^3 + 3xy$$

2. $$z = f(x, y) = \ln(x + 2y)$$

3. $$z = f(x, y) = x^4 - 6x^2y^2 + y^4$$

Example 7

Use the quotient rule to compute the partial derivatives R_x and R_y for the following function

$$R(x,y) = \frac{x^2}{y^2+1} - \frac{y^2}{x^2+y}$$

Note: we are keeping one variable constant.

Example 6(a)

Compute the partial derivatives of

$$z = f(x, y) = x^3 + 3xy.$$

$$\frac{\partial z}{\partial x} = 3x^2 + 3y$$

$$\frac{\partial z}{\partial y} = 3x$$

Example 6(b)

Compute the partial derivatives of

$$z = f(x, y) = \ln(x + 2y).$$

$$\frac{\partial z}{\partial x} = \frac{1}{x + 2y}$$
$$\frac{\partial z}{\partial y} = \frac{2}{x + 2y}$$

Example 6(c)

Compute the partial derivatives of

$$z = f(x, y) = x^4 - 6x^2y^2 + y^4.$$

$$\frac{\partial z}{\partial x} = 4x^3 - 12xy^2$$
$$\frac{\partial z}{\partial y} = -12x^2y + 4y^3$$

Example 7

Use the quotient rule to compute R_x and R_y for

$$R(x,y) = \frac{x^2}{y^2+1} - \frac{y^2}{x^2+y}.$$

Partial derivative with respect to x:

$$R_x = \frac{2x(y^2+1) - x^2(0)}{(y^2+1)^2} - \frac{0(x^2+y) - y^2(2x)}{(x^2+y)^2} = \frac{2x}{y^2+1} + \frac{2xy^2}{(x^2+y)^2}.$$

Example 7

Use the quotient rule to compute R_x and R_y for

$$R(x,y) = \frac{x^2}{y^2+1} - \frac{y^2}{x^2+y}.$$

Partial derivative with respect to x:

$$R_x = \frac{2x(y^2+1) - x^2(0)}{(y^2+1)^2} - \frac{0(x^2+y) - y^2(2x)}{(x^2+y)^2} = \frac{2x}{y^2+1} + \frac{2xy^2}{(x^2+y)^2}.$$

Partial derivative with respect to y:

$$R_y = \frac{0(y^2+1) - x^2(2y)}{(y^2+1)^2} - \frac{2y(x^2+y) - y^2(1)}{(x^2+y)^2}$$

$$R_y = -\frac{2x^2y}{(y^2+1)^2} - \frac{2y(x^2+y) - y^2}{(x^2+y)^2}.$$