MA 16020 – Applied Calculus II: Lecture 25, Power Series & Maclaurin Series

Exercise 1. Find a power series representation for

$$f(x) = \frac{10}{1 - 5x^2}.$$

Determine its radius of convergence R.

Radius of Convergence

In $f(x) = \dfrac{10}{1 - 5x^2}$, we have $r = 5x^2$. Then

$$f(x) = 10 \sum_{n=0}^{\infty} (5x^2)^n = 10 \sum_{n=0}^{\infty} 5^n x^{2n}.$$

The geometric series converges when $|r| < 1$.
Here $|5x^2| < 1 \Rightarrow |x^2| < \frac{1}{5}$.

$$\boxed{f(x) = \sum_{n=0}^{\infty} 10 \cdot 5^n x^{2n}, \quad |x| < \frac{1}{\sqrt{5}}.}$$

Exercise 2. Use a power series to approximate the definite integral using the first three terms of the series.

$$\int_0^{0.23} \frac{1}{1+x^3}\,dx$$

Use decimal notation. Round to the nearest 5 decimal places.

Exercise 3. Approximate

$$\int_0^1 \frac{1}{1+x}\,dx$$

using the first five terms of its power series expansion.

Note 1. Recall the power series:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots, \quad |x| < 1.$$

Indefinite Integral via Power Series

Recall the Power Rule:

$$\frac{d}{dx}(x^n) = nx^{n-1}, \quad \text{and} \quad \int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \quad (n \neq -1)$$

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Exercise 4. Evaluate the definite integral as a power series.

$$\int \frac{4x}{1-x^3} \, dx$$

Exercise 4 Solution

Step 1. Rewrite using geometric series:

$$\frac{1}{1-x^3} = \sum_{n=0}^{\infty} x^{3n}, \quad |x| < 1.$$

Step 2. Multiply by $4x$:

$$\frac{4x}{1-x^3} = 4\sum_{n=0}^{\infty} x^{3n+1}.$$

Step 3. Integrate term-by-term:

$$\int \frac{4x}{1-x^3}\,dx = 4\sum_{n=0}^{\infty} \int x^{3n+1}\,dx = 4\sum_{n=0}^{\infty} \frac{x^{3n+2}}{3n+2} + C$$

Final Answer:

$$\boxed{\int \frac{4x}{1-x^3}\,dx = \sum_{n=0}^{\infty} \frac{4x^{3n+2}}{3n+2} + C, \quad |x| < 1.}$$

Power Series for a Derivative

Give a power series representation for the **derivative** of the following function

$$g(x) = \frac{5x}{1 - 3x^5}$$

Hint: Don't compute the derivative yet, find the power series for $g(x)$ first.

Introduction to the Maclaurin Series

Definition: Maclaurin Series

A function $f(x)$ can be represented as a Maclaurin series if

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n,$$

provided the series converges.

Interpretation: The Maclaurin series expresses $f(x)$ as an infinite polynomial built from the function's derivatives at $x = 0$.

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Examples of Common Maclaurin Series: Special Power Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad \text{(for all } x\text{)}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \quad \text{(for all } x\text{)}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \quad \text{(for all } x\text{)}$$

Example 1. Make a table to find the Maclaurin series for

$$f(x) = \sin x$$

$$f(x) = e^x$$

Computing Maclaurin Series

Example 2. Use the Maclaurin Series for e^x, $\sin x$, and $\cos x$ to compute the Maclaurin series for the following functions.

$$f(x) = e^{-3x^2}$$

$$f(x) = \cos 4x$$

$$f(x) = x^6 e^{-2x^3}$$

Maclaurin Series Needed

Recall:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

Solutions

1. $f(x) = e^{-3x^2}$

$$e^{-3x^2} = \sum_{n=0}^{\infty} \frac{(-3x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-3)^n x^{2n}}{n!}.$$

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2. $f(x) = \cos(4x)$

$$\cos(4x) = \sum_{n=0}^{\infty} (-1)^n \frac{(4x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{16^n x^{2n}}{(2n)!}.$$

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3. $f(x) = x^6 e^{-2x^3}$

$$e^{-2x^3} = \sum_{n=0}^{\infty} \frac{(-2x^3)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-2)^n x^{3n}}{n!}.$$

Solutions

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3. $f(x) = x^6 e^{-2x^3}$

$$e^{-2x^3} = \sum_{n=0}^{\infty} \frac{(-2x^3)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-2)^n x^{3n}}{n!}.$$

$$\Rightarrow f(x) = x^6 e^{-2x^3} = x^6 \sum_{n=0}^{\infty} \frac{(-2)^n x^{3n}}{n!} = \sum_{n=0}^{\infty} \frac{(-2)^n x^{3n+6}}{n!}.$$