MA 16020 – Applied Calculus II: Lecture 24, Geometric Series & Power Series

A **geometric series** is a series of the form

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots$$

Where:

- $a =$ **first term (initial value)**
- $r =$ **common ratio**, the factor multiplied each step
- $n =$ **index of summation**, representing term number

Examples:

$$3 + 6 + 12 + 24 + \cdots = \sum_{n=0}^{\infty} 3(2)^n \quad (a = 3,\ r = 2)$$

$$5 - \frac{5}{2} + \frac{5}{4} - \frac{5}{8} + \cdots = \sum_{n=0}^{\infty} 5(-\tfrac{1}{2})^n$$

Geometric Series: Partial Sum and Convergence

For a geometric series

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$$S_N = a\,\frac{1 - r^{N+1}}{1 - r}, \quad r \neq 1.$$

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$$\lim_{N\to\infty} S_N = \begin{cases} \dfrac{a}{1-r}, & |r| < 1 \quad \text{(series converges)} \\ \text{diverges}, & |r| \geq 1 \end{cases}$$

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Example:

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1 - \frac{1}{2}} = 2.$$

Exercise 3

Directions: Compute the sum of the geometric series below. Identify the first term a and common ratio r.

$$\sum_{n=1}^{\infty} \left(-\frac{1}{2}\right)^n$$

Step 1: Rewrite in standard form

$$\sum_{n=0}^{\infty} ar^n$$

by adjusting the index and identifying the first term a and common ratio r. The standard form always starts at $n = 0$.

Exercise 3 Solution

$$\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^{n}=\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^{n+1}=\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)^{n}$$

So

$$a=-\frac{1}{2}, r=-\frac{1}{2}, \quad |r|<1$$

so

$$\frac{a}{1-r}=\frac{\frac{-1}{2}}{1+\frac{1}{2}}=-\frac{1}{3}.$$

Directions: Compute the sum of the geometric series below. Identify the first term a and common ratio r.

$$\sum_{n=0}^{\infty} 4e^{-2n}$$

Solution to Exercise 4

$$\sum_{n=0}^{\infty} 4e^{-2n} = \sum_{n=0}^{\infty} 4(e^{-2})^n$$

and

$$a = 4,\ r = e^{-2} \quad |r| = |e^{-2}| = \left|\frac{1}{e^2}\right| < 1$$

so

$$\frac{a}{1-r} = \frac{4}{1-e^{-2}}$$

.

Exercise 5

Directions: Compute the sum of the geometric series below. Identify the first term a and common ratio r.

$$\sum_{n=0}^{\infty} \frac{3^{n+2}}{4^n}$$

Solution to Exercise 5

$$\sum_{n=0}^{\infty} \frac{3^{n+2}}{4^n} = \sum_{n=0}^{\infty} \frac{3^n \cdot 3^2}{4^n} = \sum_{n=0}^{\infty} 9 \left(\frac{3}{4}\right)^n$$

so

$$a = 9, \; r = \frac{3}{4} \quad |r| < 1$$

so

$$\frac{a}{1-r} = \frac{9}{1-\frac{3}{4}} = \frac{9}{1/4} = 36$$

Exercise 6

Directions: Compute the sum of the geometric series below. Identify the first term a and common ratio r.

$$\sum_{n=1}^{\infty} \frac{3(-1)^n}{5^{2n}}$$

Solution to Exercise 6

$$\sum_{n=1}^{\infty} \frac{3(-1)^n}{5^{2n}} = \sum_{n=1}^{\infty} 3\left(\frac{-1}{25}\right)^n = 3\sum_{n=1}^{\infty}\left(\frac{-1}{25}\right)^n.$$

Rewrite starting at $n = 0$:

$$3\sum_{n=0}^{\infty}\left(\frac{-1}{25}\right)^{n+1} = 3\left(\frac{-1}{25}\right)\sum_{n=0}^{\infty}\left(\frac{-1}{25}\right)^n.$$

Thus

$$a = 3\left(\frac{-1}{25}\right) = -\frac{3}{25}, \quad r = \frac{-1}{25}.$$

Then

$$\frac{a}{1-r} = \frac{-3/25}{1+1/25} = \frac{-3/25}{26/25} = -\frac{3}{26}.$$

Power Series

A geometric series has the form

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \quad \text{if } |r| < 1.$$

A power series is very similar, but where $r = x$, a dependent variable. So a **power series** is a series of the form

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad \text{if } |x| < 1,$$

or more generally

$$\sum_{n=0}^{\infty} ax^n \quad \text{converges if } |x| < R,$$

where R is called the **radius of convergence**.

Example 1

Express the function $f(x)$ as a power series and find its radius of convergence.

1. $$f(x) = \frac{1}{1+x}.$$

2. $$g(x) = \frac{1}{3-x}$$

3. $$h(x) = \frac{2x^2}{1+x^3}$$