MA 16020 – Applied Calculus II: Lecture 23, Geometric Series

Idea: A *series* is the sum of the terms of a sequence. For example, if we have a sequence $a_1, a_2, a_3, \ldots$, then the series is written as

$$a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n.$$

Summation notation: The Greek letter Σ (sigma) means "sum."

$$\sum_{n=1}^{3} \left(\frac{2^n + 1}{3^n - 2} \right)$$

Reading the notation:

- $n = 1$: starting index — tells us where to begin.
- 3: upper limit — tells us when to stop counting.
- $\frac{2^n+1}{3^n-2}$: the general term we're summing.

Expanding the sum:

$$\frac{2^1 + 1}{3^1 - 2} + \frac{2^2 + 1}{3^2 - 2} + \frac{2^3 + 1}{3^3 - 2}.$$

Infinite Series

Definition: An *infinite series* is the sum of infinitely many terms of a sequence:

$$a_1 + a_2 + a_3 + a_4 + \cdots = \sum_{n=1}^{\infty} a_n.$$

We define the series using a **limit**.

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} a_n.$$

If this limit exists (approaches a finite number), we say the series **converges**. If the limit does not exist or grows without bound, the series **diverges.**

Examples:

$$\sum_{n=0}^{\infty} \frac{1}{n+1}, \quad \sum_{n=0}^{\infty} \frac{1}{2^n}.$$

The first one diverges (the terms shrink too slowly), while the second one converges (the terms get smaller very quickly).

Exercise 1: Partial Sums Practice

Example: Consider the infinite series

$$\sum_{n=0}^{\infty} \frac{1}{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

Find: The 3rd and 5th partial sums.

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$$S_3 = \underbrace{1 + \frac{1}{2} + \frac{1}{3}}_{\text{3 terms}} \quad = \quad \frac{11}{6} \approx 1.833.$$

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$$S_5 = \underbrace{1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}}_{\text{5 terms}} \quad = \quad \frac{137}{60} \approx 2.283.$$

Definition. An infinite series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

converges if the sequence of partial sums $S_N = \sum_{n=1}^{N} a_n$ approaches a finite limit S as $N \to \infty$.

$$\lim_{N\to\infty} S_N = S \quad \Rightarrow \quad \text{Series converges to } S.$$

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If the sequence of partial sums does **not** approach a finite limit, we say the series **diverges.**

Properties of Series

Facts about Series:

1. If c is a constant, then
$$\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$$
2. Shifting the index does not change convergence:
$$\sum_{n=0}^{\infty} a_n = \sum_{n=m}^{\infty} a_{n-m}$$
3. For any integer $m \geq 0$,
$$\sum_{n=m}^{\infty} a_n = \sum_{n=0}^{\infty} a_{n+m}$$

Exercise 2

Directions: Rewrite each series so that it begins at the indicated index.

1. Rewrite
$$\sum_{n=0}^{\infty} \frac{6^n}{n+1}$$
so that the index of summation starts at $n = 1$.

2. Rewrite
$$\sum_{n=3}^{\infty} \frac{4(3^{2n})}{5^n}$$
so that the index of summation starts at $n = 0$.

Solution 1:

$$\sum_{n=0}^{\infty} \frac{6^n}{n+1}$$

We want the index to start at $n = 1$. Using the shift property

$$\sum_{n=0}^{\infty} a_{n+1} = \sum_{n=1}^{\infty} a_n,$$

we have

$$\sum_{n=0}^{\infty} \frac{6^n}{n+1} = \sum_{n=1}^{\infty} \frac{6^{n-1}}{n}.$$

$$\boxed{\sum_{n=1}^{\infty} \frac{6^{n-1}}{n}}$$

Solution 2:

$$\sum_{n=3}^{\infty} \frac{4(3^{2n})}{5^n}$$

We want the index to start at $n = 0$. Using the shift property

$$\sum_{n=k}^{\infty} a_n = \sum_{n=0}^{\infty} a_{n+k},$$

we have

$$\sum_{n=3}^{\infty} \frac{4(3^{2n})}{5^n} = \sum_{n=0}^{\infty} \frac{4(3^{2(n+3)})}{5^{n+3}}$$

Definition: Geometric Series

A **geometric series** is a series of the form

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots$$

Where:

- $a =$ **first term (initial value)**
- $r =$ **common ratio**, the factor multiplied each step
- $n =$ **index of summation**, representing term number

Examples:

$$3 + 6 + 12 + 24 + \cdots = \sum_{n=0}^{\infty} 3(2)^n \quad (a = 3,\ r = 2)$$

$$5 - \frac{5}{2} + \frac{5}{4} - \frac{5}{8} + \cdots = \sum_{n=0}^{\infty} 5(-\tfrac{1}{2})^n$$

Geometric Series: Partial Sum and Convergence

For a geometric series

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots$$

The **partial sum of the first $N+1$ terms** is

$$S_N = a\,\frac{1 - r^{N+1}}{1 - r}, \quad r \neq 1.$$

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Convergence:

$$\lim_{N\to\infty} S_N = \begin{cases} \dfrac{a}{1-r}, & |r| < 1 \quad \text{(series converges)} \\ \text{diverges}, & |r| \geq 1 \end{cases}$$

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Example:

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1 - \frac{1}{2}} = 2.$$

Exercise 3

Directions: Compute the sum of the geometric series below. Identify the first term a and common ratio r.

$$\sum_{n=1}^{\infty} \left(-\frac{1}{2}\right)^{n}$$

Step 1: Rewrite in standard form

$$\sum_{n=0}^{\infty} ar^{n}$$

by adjusting the index and identifying the first term a and common ratio r. The standard form always starts at $n = 0$.

Exercise 3 Solution

$$\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^n = \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^{n+1} = \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)^n$$

So

$$a = -\frac{1}{2}, r = -\frac{1}{2}, \quad |r| < 1$$

so

$$\frac{a}{1-r} = \frac{\frac{-1}{2}}{1+\frac{1}{2}} = -\frac{1}{3}.$$

Directions: Compute the sum of the geometric series below. Identify the first term a and common ratio r.

$$\sum_{n=0}^{\infty} 4e^{-2n}$$

Solution to Exercise 4

$$\sum_{n=0}^{\infty} 4e^{-2n} = \sum_{n=0}^{\infty} 4(e^{-2})^n$$

and

$$a = 4,\ r = e^{-2} \quad |r| = |e^{-2}| = \left|\frac{1}{e^2}\right| < 1$$

so

$$\frac{a}{1-r} = \frac{4}{1-e^{-2}}$$

.

Exercise 5

Directions: Compute the sum of the geometric series below. Identify the first term a and common ratio r.

$$\sum_{n=0}^{\infty} \frac{3^{n+2}}{4^n}$$

Solution to Exercise 5

$$\sum_{n=0}^{\infty} \frac{3^{n+2}}{4^n} = \sum_{n=0}^{\infty} \frac{3^n \cdot 3^2}{4^n} = \sum_{n=0}^{\infty} 9 \left(\frac{3}{4}\right)^n$$

so

$$a = 9, \; r = \frac{3}{4} \quad |r| < 1$$

so

$$\frac{a}{1-r} = \frac{9}{1 - \frac{3}{4}} = \frac{9}{1/4} = 36$$

Directions: Compute the sum of the geometric series below. Identify the first term a and common ratio r.

$$\sum_{n=1}^{\infty} \frac{3(-1)^n}{5^{2n}}$$

Solution to Exercise 6

$$\sum_{n=1}^{\infty} \frac{3(-1)^n}{5^{2n}} = \sum_{n=1}^{\infty} 3\left(\frac{-1}{25}\right)^n = 3\sum_{n=1}^{\infty}\left(\frac{-1}{25}\right)^n.$$

Rewrite starting at $n = 0$:

$$3\sum_{n=0}^{\infty}\left(\frac{-1}{25}\right)^{n+1} = 3\left(\frac{-1}{25}\right)\sum_{n=0}^{\infty}\left(\frac{-1}{25}\right)^n.$$

Thus

$$a = 3\left(\frac{-1}{25}\right) = -\frac{3}{25}, \quad r = \frac{-1}{25}.$$

Then

$$\frac{a}{1-r} = \frac{-3/25}{1+1/25} = \frac{-3/25}{26/25} = -\frac{3}{26}.$$