# MA 16020 – Applied Calculus II: Lecture 22
# First Order Linear Differential Equations II

# First-Order Linear Differential Equations

**Definition:** A **first-order linear differential equation** is any equation that can be written in the form

$$y' + p(t)y = f(t),$$

where $p(t)$ and $f(t)$ are continuous on some interval $I$.

# First-Order Linear Differential Equations

**Definition:** A **first-order linear differential equation** is any equation that can be written in the form

$$y' + p(t)y = f(t),$$

where $p(t)$ and $f(t)$ are continuous on some interval $I$.

**Special Case (Homogeneous):**

$$y' + p(t)y = 0 \quad \Rightarrow \quad \text{can be solved by separation of variables.}$$

**Examples:**

- $y' + 3y = 0$ (linear & separable)
- $y' + y = e^t$ (linear, not separable)
- $y' = ty^2$ (separable, not linear)

**Key Idea:** Every first-order linear ODE can be solved systematically using an **integrating factor.**

# Integrating Factor Method (Factor is $I(t)$)

Start with the inhomogeneous linear DE

$$y' + p(t)\,y = f(t).$$

**Multiply both sides by an integrating factor $I(t)$:**

$$I(t)y' + I(t)p(t)y = I(t)f(t).$$

# Integrating Factor Method (Factor is $I(t)$)

Start with the inhomogeneous linear DE

$$y' + p(t)\,y = f(t).$$

**Multiply both sides by an integrating factor $I(t)$:**

$$I(t)y' + I(t)p(t)y = I(t)f(t).$$

**Observation:** the left-hand side will be the derivative of a product if

$$\big(I(t)y\big)' = I(t)y' + I'(t)y.$$

So we want $I'(t)y = I(t)p(t)y$ for all $y$, i.e.

$$I'(t) = p(t)\,I(t).$$

# Integrating Factor Method

**Solve for $I(t)$ (separation of variables):**

$$\frac{I'(t)}{I(t)} = p(t) \implies \int \frac{I'}{I}\,dt = \int p(t)\,dt$$

$$\ln|I(t)| = \int p(t)\,dt + C \implies I(t) = e^{\int p(t)\,dt}$$

(we may take $C = 0$ since any constant factor cancels later).

**Continue:** With this choice of $I(t)$,

$$\big(I(t)y(t)\big)' = I(t)f(t).$$

Integrate both sides:

$$I(t)y(t) = \int I(t)f(t)\,dt + C.$$

Hence the general solution is

$$\boxed{y(t) = \frac{1}{I(t)}\Big(\int I(t)f(t)\,dt + C\Big), \qquad I(t) = e^{\int p(t)\,dt}.}$$

# Mini Guide: Solving with Integrating Factors

**Goal:** Solve first-order linear DE

$$y' + p(t)\,y = f(t)$$

using an integrating factor $I(t)$.

**Steps:**

1. **Identify** $p(t)$ and $f(t)$ from the equation.

   $$\text{(must be in the form } y' + p(t)y = f(t))$$

2. **Compute the integrating factor:**

   $$I(t) = e^{\int p(t)\,dt}.$$

3. **Multiply the entire equation** by $I(t)$:

   $$I(t)y' + I(t)p(t)y = I(t)f(t).$$

4. **Recognize:** the left-hand side is $(I(t)y)' = I(t)f(t)$.
5. **Integrate both sides and solve**

**Example 1:**

$$y' + y = e^{-t}.$$

# Example 1: Inhomogeneous DE

**Example 1:**

$$y' + y = e^{-t}.$$

**Step 1:** Identify functions

$$p(t) = 1, \qquad f(t) = e^{-t}.$$

**Step 2:** Compute the integrating factor

$$I(t) = e^{\int p(t)\,dt} = e^{t}.$$

**Step 3:** Multiply through by $I(t)$

$$e^{t}y' + e^{t}y = e^{t}e^{-t} = 1.$$

**Notice:**

$$\text{LHS} = (e^{t}y)'.$$

# Example 2: Integrating Factor Practice

**Directions:** Use the integrating factor method to solve each differential equation. (Show work: identify $p(x)$, compute $I(x) = e^{\int p(x)\,dx}$, multiply through, integrate, then solve for $y$.)

1. $\dfrac{dy}{dx} + 11y = 5.$
2. $\dfrac{dy}{dx} + \dfrac{2}{x}y = 3x - 5,$ (assume $x > 0$ for the integrating factor).
3. $y' - y = 19$
4. $\dfrac{dy}{dx} + 9y = -3e^{-9x}.$

## Solution 1: $\dfrac{dy}{dx} + 11y = 5$

**Step 1: Identify.** $p(x) = 11, \; q(x) = 5.$

**Step 2: Integrating factor.**

$$I(x) = e^{\int 11\,dx} = e^{11x}.$$

**Step 3: Multiply and simplify.**

$$e^{11x}y' + 11e^{11x}y = 5e^{11x} \quad \Rightarrow \quad (e^{11x}y)' = 5e^{11x}.$$

**Step 4: Integrate.**

$$e^{11x}y = \frac{5}{11}e^{11x} + C.$$

**Step 5: Solve.**

$$\boxed{y = \frac{5}{11} + Ce^{-11x}.}$$

# Solution 2: $\dfrac{dy}{dx} + \dfrac{2}{x}y = 3x - 5,\ x > 0$

**Step 1: Identify.** $p(x) = \frac{2}{x},\ q(x) = 3x - 5.$

**Step 2: Integrating factor.**

$$I(x) = e^{\int \frac{2}{x}\,dx} = e^{2\ln x} = x^2.$$

**Step 3: Multiply through.**

$$x^2 y' + 2xy = 3x^3 - 5x^2 \quad \Rightarrow \quad (x^2 y)' = 3x^3 - 5x^2.$$

**Step 4: Integrate.**

$$x^2 y = \frac{3}{4}x^4 - \frac{5}{3}x^3 + C.$$

**Step 5: Solve.**

$$\boxed{y = \frac{3}{4}x^2 - \frac{5}{3}x + Cx^{-2}.}$$

# Solution 3: $y' - y = 19$

**Step 1: Identify.** $p(x) = -1, \; q(x) = 19.$

**Step 2: Integrating factor.**

$$I(x) = e^{\int -1\,dx} = e^{-x}.$$

**Step 3: Multiply through.**

$$e^{-x}y' - e^{-x}y = 19e^{-x} \quad \Rightarrow \quad (e^{-x}y)' = 19e^{-x}.$$

**Step 4: Integrate.**

$$e^{-x}y = -19e^{-x} + C.$$

**Step 5: Solve.**

$$\boxed{y = Ce^{x} - 19.}$$

# Solution 4: $\frac{dy}{dx} + 9y = -3e^{-9x}$

**Step 1: Identify.** $p(x) = 9, \; q(x) = -3e^{-9x}$.

**Step 2: Integrating factor.**

$$I(x) = e^{\int 9\,dx} = e^{9x}.$$

**Step 3: Multiply through.**

$$e^{9x}y' + 9e^{9x}y = -3e^{9x}e^{-9x} = -3 \quad \Rightarrow \quad (e^{9x}y)' = -3.$$

**Step 4: Integrate.**

$$e^{9x}y = -3x + C.$$

**Step 5: Solve.**

$$\boxed{y = e^{-9x}(C - 3x).}$$

# Example 3: More Integrating Factor Practice

**Directions:** Use the integrating factor method to solve each differential equation. (Show work: identify $p(x)$, compute $I(x) = e^{\int p(x)\,dx}$, multiply through, integrate, then solve for $y$.)

1. $\dfrac{dy}{dx} + \frac{3y}{x} = \frac{e^x}{x^3}$.
2. $\dfrac{dy}{dx} - \frac{3y}{x+1} = (x+1)^4$.
3. $3x^2y + x^3\dfrac{dy}{dx} = 7\sec^2 x \tan x$.
4. $x^4y' + x^3y = 4x^4\cos x$

## Solution 1: $\dfrac{dy}{dx} + \dfrac{3y}{x} = \dfrac{e^x}{x^3}$

**Step 1: Identify.** $p(x) = \dfrac{3}{x}, \quad q(x) = \dfrac{e^x}{x^3}.$

**Step 2: Integrating factor.**

$$I(x) = e^{\int \frac{3}{x}\,dx} = e^{3\ln x} = x^3 \quad (x > 0).$$

**Step 3: Multiply through.**

$$x^3 y' + 3x^2 y = e^x \quad \Rightarrow \quad (x^3 y)' = e^x.$$

**Step 4: Integrate.**

$$x^3 y = e^x + C.$$

**Step 5: Solve.**

$$\boxed{y = \frac{e^x + C}{x^3}.}$$

## Solution 2: $\frac{dy}{dx} - \frac{3y}{x+1} = (x+1)^4$

**Step 1: Identify.** $p(x) = -\frac{3}{x+1}, \quad q(x) = (x+1)^4.$

**Step 2: Integrating factor.**

$$I(x) = e^{\int -\frac{3}{x+1}\,dx} = e^{-3\ln|x+1|} = (x+1)^{-3}.$$

**Step 3: Multiply through.**

$$(x+1)^{-3}y' - \frac{3y}{x+1}(x+1)^{-3} = (x+1)^4(x+1)^{-3} = (x+1).$$

$$\Rightarrow ((x+1)^{-3}y)' = (x+1).$$

**Step 4: Integrate.**

$$(x+1)^{-3}y = \int (x+1)\,dx = \frac{(x+1)^2}{2} + C.$$

**Step 5: Solve.**

# Solution 3: $3x^2y + x^3\dfrac{dy}{dx} = 7\sec^2 x \tan x$

**Step 1: Recognize.** The left-hand side looks like the derivative of $x^3y$:

$$(x^3y)' = 3x^2y + x^3y'.$$

**Step 2: Simplify.**

$$(x^3y)' = 7\sec^2 x \tan x.$$

**Step 3: Integrate both sides.**

$$x^3y = \int 7\sec^2 x \tan x\, dx.$$

Let $u = \tan x \Rightarrow du = \sec^2 x\, dx$.

$$x^3y = 7\int u\, du = \frac{7}{2}u^2 + C = \frac{7}{2}\tan^2 x + C.$$

**Step 4: Solve.**

$$\boxed{y = \frac{7}{2}x^{-3}\tan^2 x + Cx^{-3}.}$$

# Solution 4: $x^4 y' + x^3 y = 4x^4 \cos x$

**Step 1: Standard form.** Divide through by $x^4$ ($x > 0$):

$$y' + \frac{y}{x} = 4\cos x.$$

**Step 2: Integrating factor.**

$$I(x) = e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x.$$

**Step 3: Multiply through.**

$$xy' + y = 4x\cos x \quad \Rightarrow \quad (xy)' = 4x\cos x.$$

**Step 4: Integrate.**

$$xy = 4\int x\cos x\,dx.$$

Integrate by parts: $u = x,\ dv = \cos x\,dx \Rightarrow du = dx,\ v = \sin x$. So

$$xy = 4(x\sin x + \cos x) + C.$$

**Step 5: Solve.**

# Example 4: Initial Value Problem

Ex.4: Solve the initial value problem (IVP) given by

$$\frac{dy}{dx} = y \tan x - \sec x; \quad y(0) = 1.$$

## Solution Ex 4 (IVP): $\frac{dy}{dx} = y \tan x - \sec x, \;\; y(0) = 1$

**Step 1: Standard form.**

$$y' - y \tan x = -\sec x.$$

Hence $p(x) = -\tan x, \;\; q(x) = -\sec x.$

**Step 2: Integrating factor.**

$$I(x) = e^{\int -\tan x \, dx} = e^{\ln(\cos x)} = \cos x.$$

**Step 3: Multiply through.**

$$\cos x \, y' - y \sin x = -1 \quad \Rightarrow \quad (\cos x \, y)' = -1.$$

**Step 4: Integrate.**

$$\cos x \, y = -x + C.$$

**Step 5: Apply initial condition.**

$$1 = C \Rightarrow \boxed{y = \frac{1 - x}{\cos x}.}$$