MA 16020 – Applied Calculus II: Lecture 21
First Order Linear Differential Equations I

First-Order Linear Differential Equations

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where $p(t)$ and $f(t)$ are continuous on some interval I.

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- $y' + y = e^t$ (linear, not separable)
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Key Idea: Every first-order linear ODE can be solved systematically using an **integrating factor.**

What Makes an ODE *Linear*?

General Form:

$$\boxed{y' + p(t)y = f(t)}$$

A first-order ODE is linear if:

- y and y' appear only to the **first power**.
- y and y' are **not multiplied together**.
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Examples:

$y' + 3y = 0$	✓ Linear (homogeneous)
$y' + y = e^t$	✓ Linear
$y' = ty^2$	× Nonlinear (power of y)
$y' + \sin(y) = t$	× Nonlinear (nonlinear function of y)
$y' + yy' = t$	× Nonlinear (product yy')

Summary: Linear = y' and y appear "plain," like coefficients in a line equation — no powers, products, or nonlinear functions.

Exercise: Linear or Nonlinear, Separable or Not?

Determine for each first-order ODE:

- **Linear or Nonlinear**
- **Separable or Not Separable**

Equations:

1. $y' + 5y = \cos t$
2. $y' = y^2 \sin t$
3. $y' - \frac{3}{t} y = t^2$
4. $y' = e^t \sqrt{y}$
5. $y' + \tan t \, y^2 = \sec t$

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Answers:

$y' + 5y = \cos t$	✓ Linear	× Not separable
$y' = y^2 \sin t$	× Nonlinear	✓ Separable
$y' - \frac{3}{t} y = t^2$	✓ Linear	× Not separable
$y' = e^t \sqrt{y}$	× Nonlinear	✓ Separable
$y' + \tan t \, y^2 = \sec t$	× NonLinear	× Not separable

First-Order Linear Differential Equations

Motivation: A simple but very important type of separable equation is the **first-order homogeneous linear differential equation**.

Definition 5.21: First-Order Homogeneous Linear DE

A **first-order homogeneous linear differential equation** is an equation of the form

$$y' + p(t)y = 0,$$

or equivalently,

$$y' = -p(t)y,$$

where $p(t)$ is a continuous function on some interval.

Key idea: These equations are **separable** and can be solved by standard techniques for separable equations, making them foundational for more general first-order linear DEs.

Solving First-Order Homogeneous Linear DEs

Recap: We know separation of variables:

$$y' = -p(t)y \quad \Rightarrow \quad \frac{dy}{y} = -p(t)\,dt.$$

Step 1: Integrate both sides.

$$\int \frac{1}{y}\,dy = -\int p(t)\,dt \quad \Rightarrow \quad \ln|y| = -\int p(t)\,dt + C.$$

Step 2: Solve for $y(t)$.

$$y(t) = Ce^{-\int p(t)\,dt}, \quad C \text{ is an arbitrary constant.}$$

Remark: This formula gives a **quick solution method** for any first-order homogeneous linear DE without repeated separation of variables steps.

Example 1: Solve a First-Order Homogeneous Linear DE (Part a)

Problem: Solve the initial value problem

$$y' + y\cos t = 0$$

subject to $y(0) = \frac{1}{2}$.

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Problem: Solve the initial value problem

$$y' + y\cos t = 0$$

subject to $y(0) = \frac{1}{2}$. **Step 1: Identify $p(t)$.** Here, $p(t) = \cos t$.

Step 2: Apply the general formula for homogeneous linear DEs.

$$y(t) = Ce^{-\int p(t)dt} = Ce^{-\int \cos t\, dt} = Ce^{-\sin t}.$$

Step 3: Apply initial condition $y(0) = \frac{1}{2}$.

$$y(0) = Ce^{-\sin 0} = Ce^{0} = C = \frac{1}{2}.$$

Solution:

$$\boxed{y(t) = \frac{1}{2}e^{-\sin t}}$$

Example 1: Solve a First-Order Homogeneous Linear DE (Part b)

Problem: Solve

$$y' + y\cos t = 0$$

subject to $y(2) = \frac{1}{2}$.

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Problem: Solve

$$y' + y\cos t = 0$$

subject to $y(2) = \frac{1}{2}$. **Step 1: General solution.**

$$y(t) = Ce^{-\sin t}.$$

Step 2: Apply initial condition $y(2) = \frac{1}{2}$**.**

$$\frac{1}{2} = Ce^{-\sin 2} \quad \Rightarrow \quad C = \frac{1}{2}e^{\sin 2}.$$

Solution:

$$\boxed{y(t) = \frac{1}{2}e^{\sin 2}\, e^{-\sin t} = \frac{1}{2}e^{\sin 2 - \sin t}}$$

Example 2: First-Order Homogeneous Linear DE (Pause Before Solution)

Problem: Solve the initial value problem

$$ty' + 3y = 0, \quad y(1) = 2,$$

assuming $t > 0$.

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Problem: Solve the initial value problem

$$ty' + 3y = 0, \quad y(1) = 2,$$

assuming $t > 0$. **Step 1: Rewrite in standard form.** Divide through by t (since $t > 0$):

$$y' + \frac{3}{t}y = 0.$$

At this point, notice this is a first-order homogeneous linear DE with

$$p(t) = \frac{3}{t}.$$

Think about applying the general solution formula $y(t) = Ce^{-\int p(t)dt}$ before continuing.

Try solving the following initial value problems using the formula $y(t) = Ce^{-\int p(t)\,dt}$:

Example 3:

$$y' + 2t\,y = 0, \quad y(0) = 3$$

Example 4:

$$y' + \sin t\,y = 0, \quad y\left(\frac{\pi}{2}\right) = 4$$

Instructions:

- Identify $p(t)$ in each DE.
- Apply the general solution formula for homogeneous linear DEs: $y(t) = Ce^{-\int p(t)\,dt}$.
- Solve for the constant using the initial condition.

Practice Examples: Quick Solutions

Example 3: $y' + 2t\,y = 0,\ y(0) = 3$

$$y(t) = Ce^{-\int 2t\,dt} = Ce^{-t^2}, \quad y(0) = 3 \Rightarrow C = 3$$

$$\boxed{y(t) = 3e^{-t^2}}$$

Example 4: $y' + \sin t\,y = 0,\ y(\pi/2) = 4$

$$y(t) = Ce^{-\int \sin t\,dt} = Ce^{\cos t}, \quad y(\pi/2) = 4 \Rightarrow C = 4e^{-\cos(\pi/2)} = 4$$

$$\boxed{y(t) = 4e^{\cos t}}$$

First-Order Inhomogeneous Linear DEs

Definition 5.22. A **first-order inhomogeneous linear differential equation** is one of the form

$$y' + p(t)y = f(t),$$

where $p(t)$ and $f(t)$ are continuous functions on some interval I.

Observation: When $f(t) = 0$, the equation is *homogeneous*. When $f(t) \neq 0$, the equation is *inhomogeneous*.

Examples:

1. $y' + y = e^t$
2. $y' + 2y = 5t$
3. $y' + y \tan t = \sin t$

The goal: find a systematic way to solve these — introducing the **integrating factor**.

Integrating Factor Method (Factor is $I(t)$)

Start with the inhomogeneous linear DE

$$y' + p(t)\, y = f(t).$$

Multiply both sides by an integrating factor $I(t)$:

$$I(t)y' + I(t)p(t)y = I(t)f(t).$$

Integrating Factor Method (Factor is $I(t)$)

Start with the inhomogeneous linear DE

$$y' + p(t)\,y = f(t).$$

Multiply both sides by an integrating factor $I(t)$:

$$I(t)y' + I(t)p(t)y = I(t)f(t).$$

Observation: the left-hand side will be the derivative of a product if

$$\big(I(t)y\big)' = I(t)y' + I'(t)y.$$

So we want $I'(t)y = I(t)p(t)y$ for all y, i.e.

$$I'(t) = p(t)\,I(t).$$

Integrating Factor Method

Solve for $I(t)$ (separation of variables):

$$\frac{I'(t)}{I(t)} = p(t) \quad \Longrightarrow \quad \int \frac{I'}{I}\,dt = \int p(t)\,dt$$

$$\ln|I(t)| = \int p(t)\,dt + C \quad \Longrightarrow \quad I(t) = e^{\int p(t)\,dt}$$

(we may take $C = 0$ since any constant factor cancels later).

Continue: With this choice of $I(t)$,

$$\big(I(t)y(t)\big)' = I(t)f(t).$$

Integrate both sides:

$$I(t)y(t) = \int I(t)f(t)\,dt + C.$$

Hence the general solution is

$$\boxed{y(t) = \frac{1}{I(t)}\Big(\int I(t)f(t)\,dt + C\Big), \qquad I(t) = e^{\int p(t)\,dt}.}$$

Mini Guide: Solving with Integrating Factors

Goal: Solve first-order linear DE

$$y' + p(t)\,y = f(t)$$

using an integrating factor $I(t)$.

Steps:

1. **Identify** $p(t)$ and $f(t)$ from the equation.

   (must be in the form $y' + p(t)y = f(t)$)

2. **Compute the integrating factor:**

   $$I(t) = e^{\int p(t)\,dt}.$$

3. **Multiply the entire equation** by $I(t)$:

   $$I(t)y' + I(t)p(t)y = I(t)f(t).$$

4. **Recognize:** the left-hand side is $(I(t)y)' = I(t)f(t)$.
5. **Integrate both sides and solve**

New Example 1: Inhomogeneous DE

Example 1:

$$y' + y = e^{-t}.$$

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$$y' + y = e^{-t}.$$

Step 1: Identify functions

$$p(t) = 1, \qquad f(t) = e^{-t}.$$

Step 2: Compute the integrating factor

$$I(t) = e^{\int p(t)\,dt} = e^{t}.$$

Step 3: Multiply through by $I(t)$

$$e^{t}y' + e^{t}y = e^{t}e^{-t} = 1.$$

Notice:

$$\text{LHS} = (e^{t}y)'.$$

New Example 2: Integrating Factor Practice

Directions: Use the integrating factor method to solve each differential equation. (Show work: identify $p(x)$, compute $I(x) = e^{\int p(x)\,dx}$, multiply through, integrate, then solve for y.)

1. $\dfrac{dy}{dx} + 11y = 5.$
2. $\dfrac{dy}{dx} + \dfrac{2}{x}y = 3x - 5,$ (assume $x > 0$ for the integrating factor).
3. $\dfrac{dy}{dx} + 9y = -3e^{-9x}.$