MA 16020 – Applied Calculus II: Lecture 20
Separable Differential Equations II: Continued

Example 1: General Solution

Example 1. Find the **general solution** to the differential equation

$$\frac{dy}{dt} = k(50 - y),$$

assuming $50 - y > 0$.

Example 2: Radioactive Decay and Half-Life

Suppose $P(t)$ is the mass of a radioactive substance at time t. The differential equation is

$$P'(t) = -\frac{3}{2}P(t).$$

Instructions:

- Find the **half-life** t such that $P(t) = \frac{A}{2}$ where A is the amount at time 0.

Half-Life Definition

The **half-life** of a substance is the amount of time it takes for half the substance to disappear.

Solution: Example 2 (Radioactive Decay and Half-Life)

We are given

$$P'(t) = -\frac{3}{2}P(t), \quad P(0) = A.$$

Step 1: Solve the differential equation.

$$\frac{P'(t)}{P(t)} = -\frac{3}{2} \quad \Longrightarrow \quad \int \frac{P'(t)}{P(t)}\,dt = \int -\frac{3}{2}\,dt$$

$$\ln P(t) = -\frac{3}{2}t + C \quad \Longrightarrow \quad P(t) = Ae^{-\frac{3}{2}t}.$$

Step 2: Find the half-life.

$$P(t) = \frac{A}{2} \quad \Rightarrow \quad Ae^{-\frac{3}{2}t} = \frac{A}{2}.$$

$$e^{-\frac{3}{2}t} = \frac{1}{2} \quad \Rightarrow \quad t = \frac{2}{3}\ln 2.$$

Therefore, the half-life is:

$$\boxed{t_{1/2} = \frac{2}{3}\ln 2.}$$

Newton's Law of Cooling: Intuition

Idea: The rate at which an object cools (or warms) is **proportional to the temperature difference** between the object and its surroundings.

If $T(t)$ is the temperature of the object at time t, and S is the surrounding (ambient) temperature, then:

$$\frac{dT}{dt} \propto (S - T).$$

That is,

$$\frac{dT}{dt} = k(S - T),$$

where k is a positive constant depending on how easily the object exchanges heat with its surroundings.

Example 3: Newton's Law of Cooling

Problem: Suppose a pot roast was 175°F when removed from the oven and placed in a 70°F room. After 10 minutes, the pot roast's temperature is 160°F.

Question: What will the temperature of the pot roast be after one hour? (Round your answer to four decimal places.)

Step 1: Identify given information.

$$T(0) = 175, \quad S = 70, \quad T(10) = 160.$$

Find $T(60)$.

Model:

$$\frac{dT}{dt} = k(S - T),$$

where $k > 0$ is the cooling constant.

Problem: After 10 minutes in Jean-Luc's room, his tea has cooled from 100°C to 45°C. The room temperature is 25°C.

Question: How much longer will it take for the tea to cool to 37°C? (Round your answer to the nearest hundredth of a minute.)

Step 1: Identify given information.

$$T(0) = 100, \quad S = 25, \quad T(10) = 45.$$

Find t such that $T(t) = 37$.

Model:

$$\frac{dT}{dt} = k(S - T),$$

where $k > 0$ is the cooling constant.

Example 5: Inflating Balloon

Problem: The volume $V(t)$ of a balloon being inflated satisfies

$$\frac{dV}{dt} = 10\,\sqrt[5]{V^2},$$

where t is time in seconds after the balloon begins to inflate. The balloon pops when $V = 400\ \text{cm}^3$.

Question: After how many seconds will the balloon pop? (Round your answer to three decimal places.)

Step 1: Identify given information.

$$\frac{dV}{dt} = 10\,(V^2)^{1/5} = 10\,V^{2/5}, \qquad V_{\text{pop}} = 400.$$

$$\text{Find } t \text{ such that } V(t) = 400.$$

Note: If not provided, state the assumed initial volume (commonly $V(0) = 0$ if appropriate).

Example 6: Chemical Reaction

Problem: During a chemical reaction, a substance is converted into a different substance at a rate **inversely proportional** to the amount of the original substance at any given time t.

Initially, there are 10 grams of the original substance. After one hour, only 8 grams remain.

Question: How much of the original substance remains after 2 hours?

Step 1: Identify given information.

$$\frac{dA}{dt} \propto \frac{1}{A} \quad \Longrightarrow \quad \frac{dA}{dt} = \frac{k}{A},$$

where $A(t)$ is the amount of the original substance at time t.

$$A(0) = 10, \quad A(1) = 8.$$

Find $A(2)$.

Example 7: Drying Towel

Problem: A wet towel hung on a clothesline to dry outside loses moisture at a rate **proportional to its moisture content**. After 1 hour, the towel has lost 32% of its original moisture content.

Question: After how long will the towel have lost 74% of its original moisture content?

Step 1: Identify given information.

$$\frac{dM}{dt} = -kM, \quad M(0) = M_0.$$

After one hour:

$$M(1) = 0.68M_0.$$

We want to find the time t when:

$$M(t) = 0.26M_0.$$

Example 7: Drying Towel (Solution)

Model:

$$\frac{dM}{dt} = -kM \quad \Longrightarrow \quad M(t) = M_0 e^{-kt}.$$

Step 1: Use given data to find k.

$$M(1) = 0.68M_0 \quad \Rightarrow \quad 0.68 = e^{-k}.$$

$$k = -\ln(0.68) \approx 0.3857.$$

Step 2: Find the time when $M(t) = 0.26M_0$.

$$0.26 = e^{-0.3857t}.$$

$$t = \frac{-\ln(0.26)}{0.3857} \approx 3.55.$$

Therefore, the towel will have lost 74% of its moisture after approximately:

$$\boxed{t = 3.55 \text{ hours.}}$$

Example 8: Snowplow and Snow Depth

Problem: The rate of change in the number of miles of road cleared per hour by a snowplow with respect to the depth of the snow is **inversely proportional** to the depth of the snow.

Given that:

$$R = 24 \text{ miles/hour when } D = 2.1 \text{ inches},$$

$$R = 13 \text{ miles/hour when } D = 8 \text{ inches}.$$

Question: How many miles of road will be cleared each hour when the depth of the snow is 13 inches?

Step 1: Identify given information.

$$\frac{dR}{dD} \propto \frac{1}{D} \quad \Longrightarrow \quad \frac{dR}{dD} = \frac{k}{D},$$

where R is the clearing rate and D is the snow depth.

Example 8: Snowplow and Snow Depth (Solution)

Model:

$$\frac{dR}{dD} = \frac{k}{D}.$$

Step 1: Integrate both sides.

$$\int dR = k \int \frac{1}{D}\, dD \quad \Longrightarrow \quad R = k \ln D + C.$$

Step 2: Use given data to find k and C.

$$\begin{cases} 24 = k \ln(2.1) + C, \\ 13 = k \ln(8) + C. \end{cases}$$

Subtract:

$$11 = k[\ln(2.1) - \ln(8)] = k \ln\left(\frac{2.1}{8}\right).$$

$$k = \frac{11}{\ln(2.1/8)} \approx -7.614.$$

Example 8 Solution Continued

Substitute to find C:

$$24 = (-7.614)\ln(2.1) + C \quad \Rightarrow \quad C \approx 29.66.$$

Step 3: Find R when $D = 13$.

$$R = -7.614\ln(13) + 29.66 \approx 11.75.$$

Therefore,

$$\boxed{R(13) \approx 11.75 \text{ miles per hour.}}$$