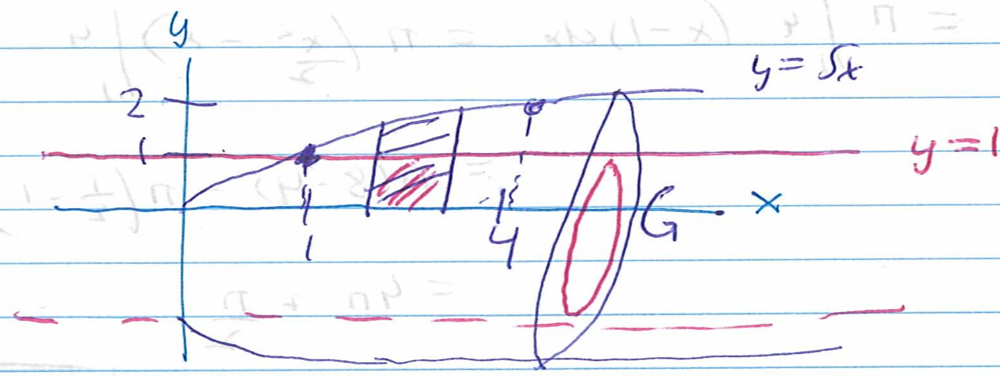
Lesson 15: Solids of Revolution: Washer Method

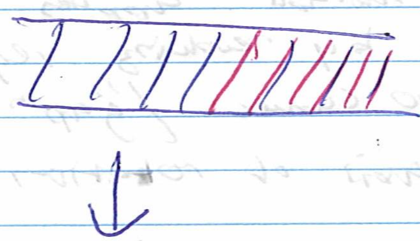
We continue computing volumes of solids of revolution. Let's consider an example with a revolving region between two curves.

Ex 1 What is the volume of the solid formed by revolving region between $y = \sqrt{x}$ (blue) and $y = 1$ over $[1, 4]$ about the x-axis? (pink)



x	$y = \sqrt{x}$
0	0
1	1
4	2

The cross section here changes from a disk to a washer



$\boxed{\text{area} = \pi R^2 - \pi r^2}$

Washer

Thickened washer

Volume $= (\pi R^2 - \pi r^2)dx$

$R = \sqrt{x} - 0 = \sqrt{x}$
$r = 1 - 0 = 1$

$\therefore$

$$\text{Volume} = \int_1^4 \pi \left((\sqrt{x})^2 - 1^2 \right) dx$$

$$= \pi \int_1^4 (x-1)\, dx = \pi \left(\frac{x^2}{2} - x \right) \Big|_1^4$$

$$= \pi(8-4) - \pi\left(\frac{1}{2} - 1\right)$$

$$= 4\pi + \frac{\pi}{2}$$

$$= \frac{8\pi + \pi}{2} = \boxed{\frac{9\pi}{2}}$$

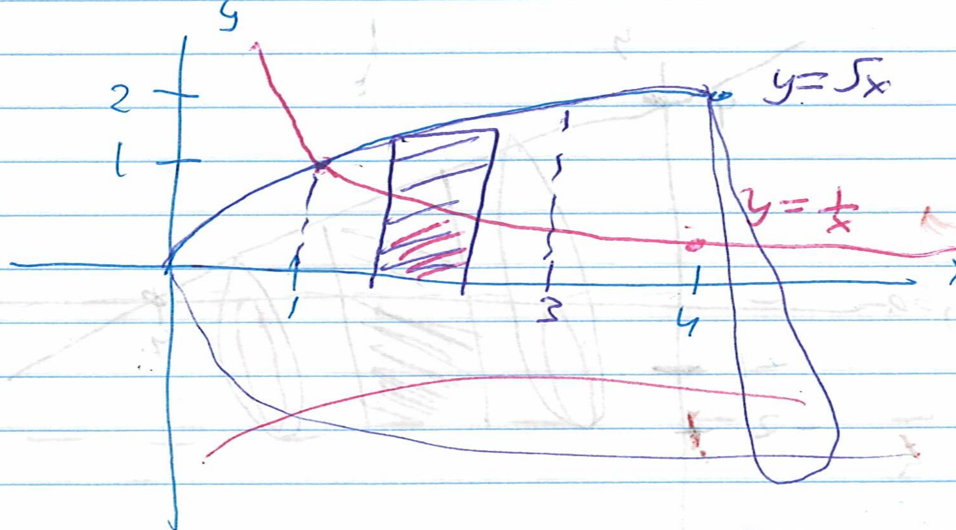
☆ Washer method applies to solids formed by revolving region between two curves / "gap" between region & axis of rotation.

Washer Method

If $f(x)$, $g(x)$ functions with $f(x) \geq g(x)$ for x in $[a,b]$, then volume of solid formed by revolving region between $f(x)$ & $g(x)$ over $[a,b]$ about x-axis is

$$\boxed{\int_a^b \pi \left((f(x))^2 - (g(x))^2 \right) dx}$$

Ex. 3 Find volume of solid formed by revolving region bounded by $\sqrt{x}$ and $\frac{1}{x}$ over interval $[1,3]$ about x-axis.



x	$1/x$
1	1
4	1/4

Washer

$$r(x) = \frac{1}{x} - 0 = \frac{1}{x}$$

$$R(x) = \sqrt{x} - 0 = \sqrt{x}$$

$$\text{Volume} = \int_1^3 \pi\left((\sqrt{x})^2 - \left(\frac{1}{x}\right)^2\right)dx$$

$$= \pi\int_1^3 \left(x - \frac{1}{x^2}\right)dx$$

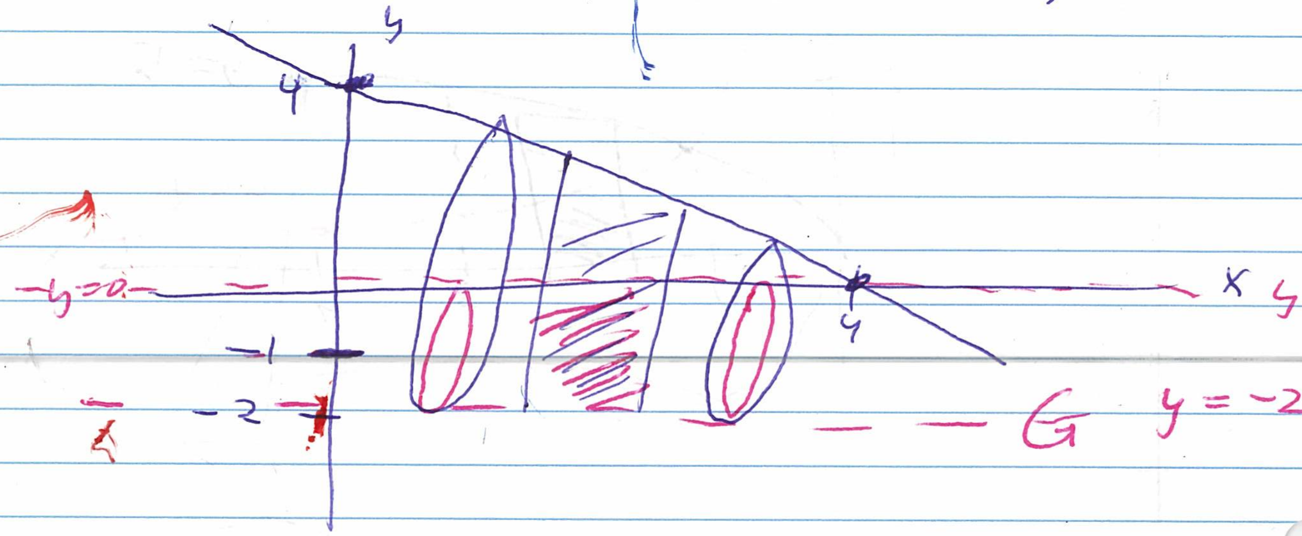
$$= \pi\left(\frac{x^2}{2} + x^{-1}\right)\Big|_1^3$$

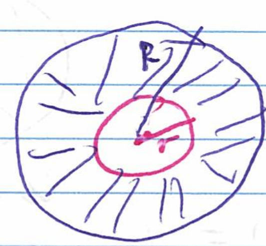
$$= \pi\left(\frac{9}{2} + \frac{1}{3} - \frac{1}{2} - 1\right)$$

$$= \pi\left(4 - 1 + \frac{1}{3}\right) = \pi\left(3 + \frac{1}{3}\right) = \boxed{\frac{10}{3}\pi}$$

Ex. 3. Find the volume of solid formed by revolving region bounded by $4-x$ and x-axis about the line $y=-2$ over $[0,4]$

This time we revolve about line $y=-2$.





Washer since the line is $y=-2$ for rotation

$R(x) = (4-x) - (-2) = 4-x+2 = 6-x$

$r(x) = 0-(-2) = 2$

$$\text{Volume} = \int_0^4 \pi\left((6-x)^2 - 2^2\right)dx$$

$$= \pi\int_0^4 (36 - 12x + x^2 - 4)\,dx$$

$$= \pi\int_0^4 (32 - 12x + x^2)\,dx$$

$$= \pi \left[32x - \frac{12x^2}{2} + \frac{x^3}{3} \right] \Big|_0^4$$

$$= \pi \left[32x - 6x^2 + \frac{x^3}{3} \right] \Big|_0^4$$

$$= \pi \left[32(4) - 6(4)^2 + \frac{4^3}{3} - 0 \right]$$

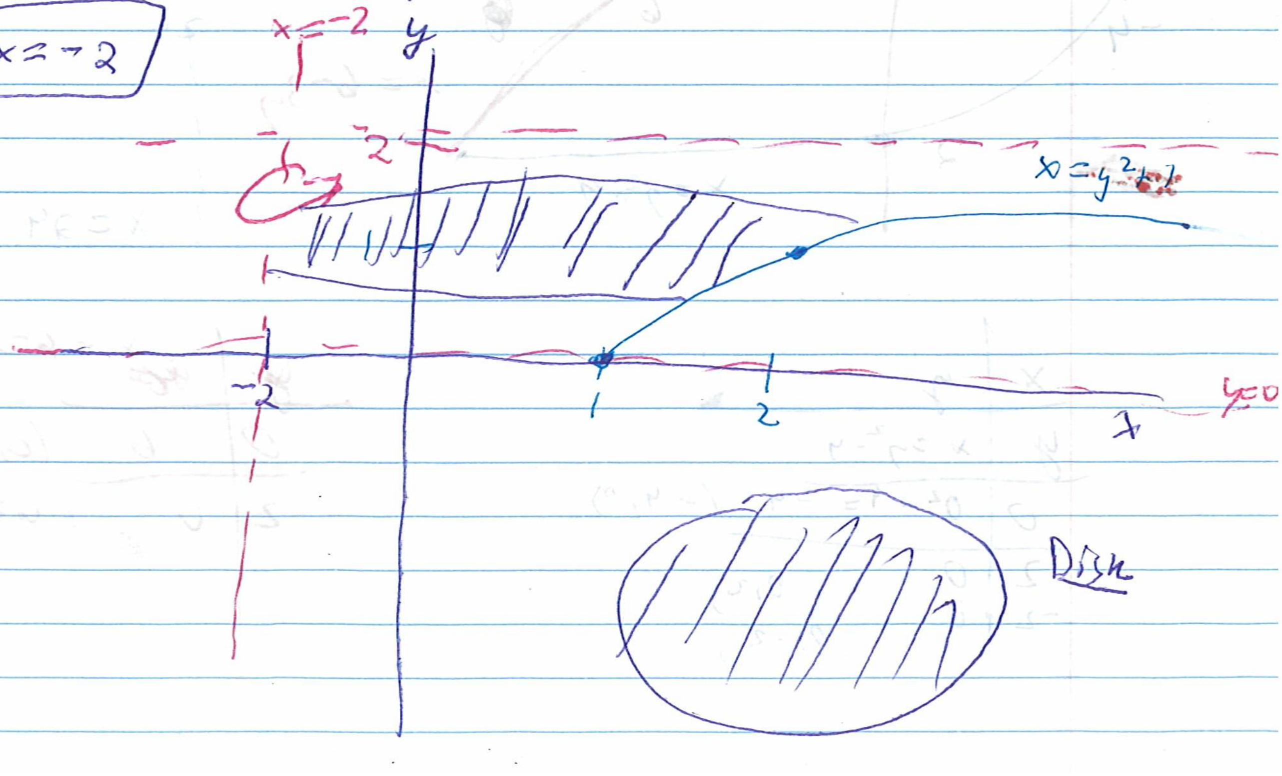
$$= \pi \left[128 - 96 + \frac{64}{3} \right]$$

$$= \pi \left[32 + \frac{64}{3} \right]$$

$$= \boxed{\frac{160\pi}{3}}$$

Ex. 4 Find the volume of the solid that results when region enclosed by $x = y^2 + 1$, $x = -2$, $y = 0$, $y = 2$ about $\boxed{x = -2}$

y	$x = y^2+1$	
0	1	(1, 0)
1	2	(2, 1)
-1	2	



Disk

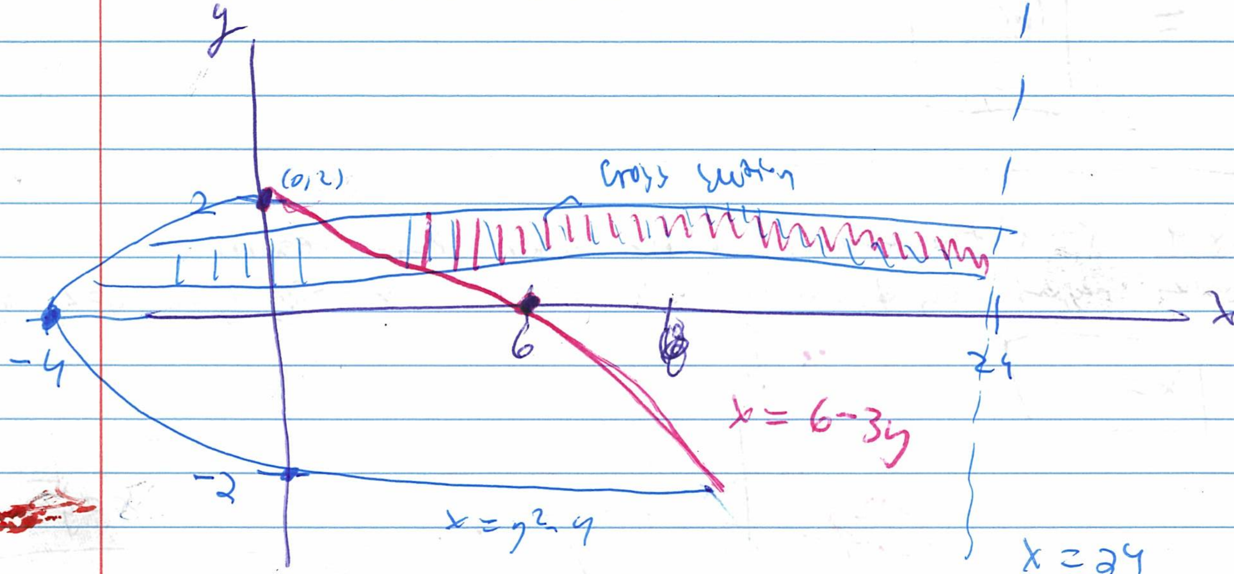
$R(y) = (y^2+1) - (-2) = y^2+3$

Disk method: Volume $= \pi \int_0^2 (y^2+3)^2 \, dy$

$= \pi \int_0^2 y^4 + 6y^2 + 9 \, dy = \boxed{\frac{202}{5}\pi} \approx 126.92$

Ex. 8 Determine volume of the solid obtained by rotating region bounded by

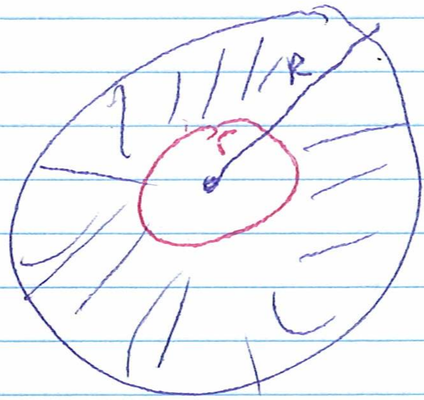
(Blue) $x = y^2 - 4$ and $x = 6 - 3y$ (pink) about the line $x = 24$.



x	y	
y	$x = y^2-4$	
0	$0^2 - 4 = -4$	$(-4,0)$
2	0	$(0,2)$
-2	0	$(0,-2)$

$x = 6-3y$

y	x	
0	6	$(6,0)$
2	0	$(0,2)$



Washer

outer radius $R(y) = 24 - (y^2 - 4) = 28 - y^2$

inner radius : $r(y) = 24 - (6 - 3y) = 18 + 3y$

intersection points: $y^2 - 4 = 6 - 3y$

$\iff y^2 + 3y - 10 = 0$

$(y+5)(y-2) = 0$

$\Rightarrow y = -5 \; , \; y = 2$

$$\text{Volume} = \pi \int_{-5}^{2} \left[(28 - y^2)^2 - (18 + 3y)^2 \right] dy$$

$$= \pi \int_{-5}^{2} (460 - 108y - 65y^2 + y^4)\, dy$$

$$= \pi \left(460y - 54y^2 - \frac{65}{3} y^3 + \frac{1}{5} y^5 \right) \Big|_{-5}^{2}$$

$$\boxed{= \frac{31556}{15} \pi}$$