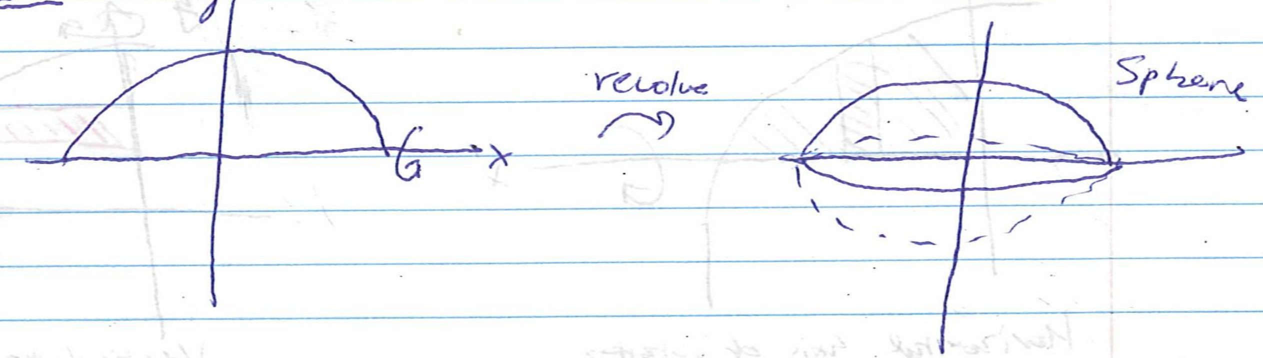
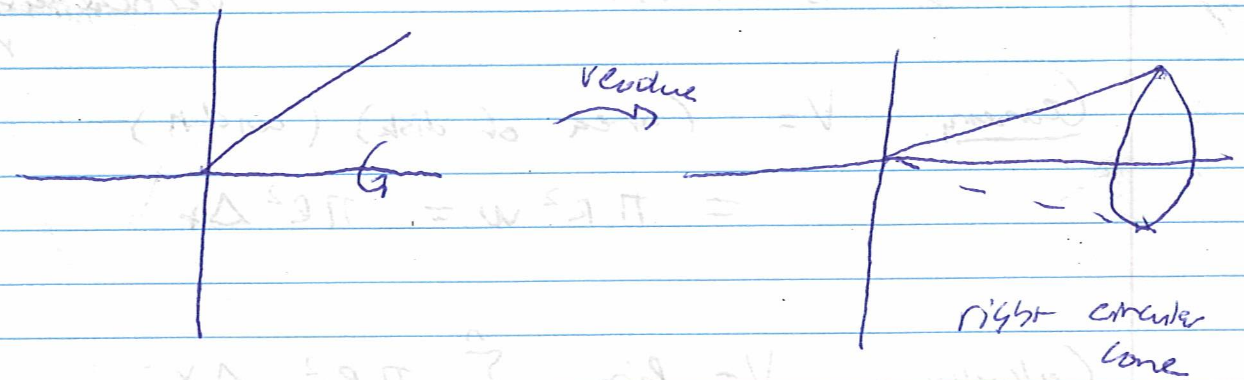
Lesson 14 - Solids of Revolution - Disk Method    10/01

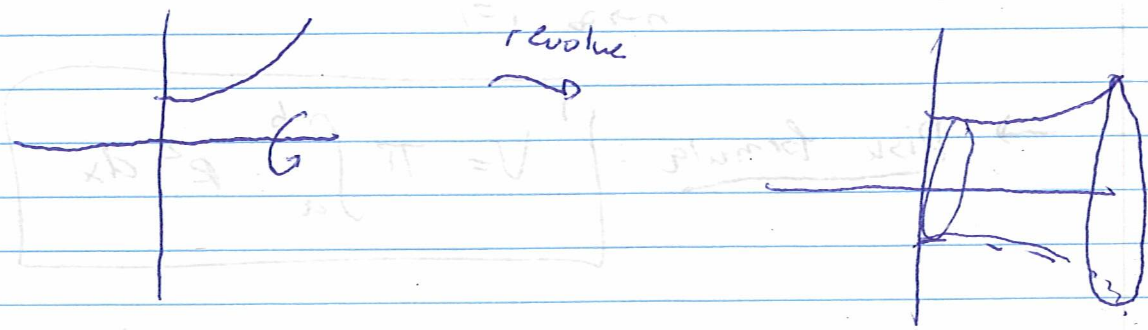
Will use what we learned about area btw. curves to find volume of solids of revolution!

$f$ continuous on interval $[a,b]$. If area between the graphs of $f$ and $x$-axis on $[a,b]$ rotated about $x$-axis, then a solid of revolution is generated:

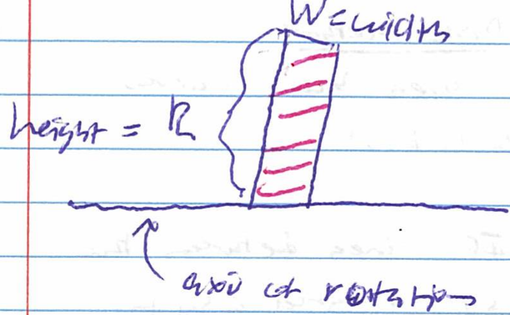
Ex. 1   rotate semicircle about $x$-axis

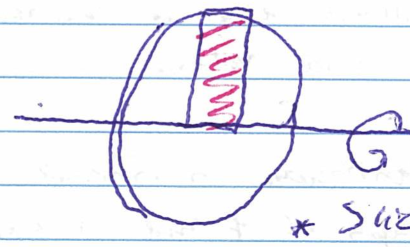


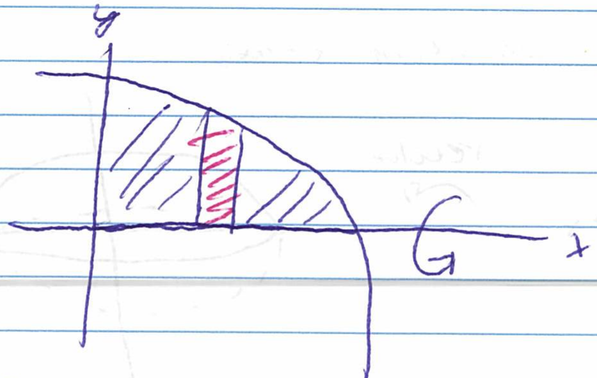




Goal: take a 2D region, revolve it around a horizontal or vertical line to create a 3D solid.

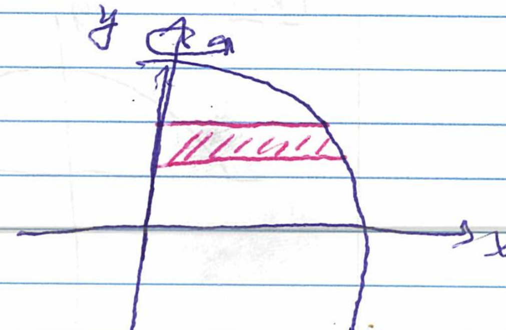


* Slice is $\perp$ to axis of rotation




Horizontal axis of rotation




Vertical axis of rotation

Geometry. $V =$ (area of disk) (width)

$$= \pi R^2 w = \pi R^2 \Delta x$$

Calculus. $V = \lim_{n \to \infty} \sum_{i=1}^{n} \pi R_x^2 \, \Delta x_i$

→ Disk formula : $\boxed{V = \pi \int_a^b R^2 \, dx}$

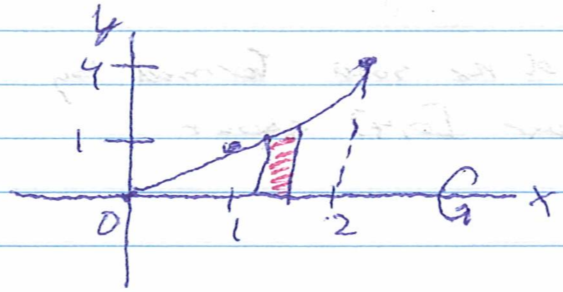
Horizontal Axis of Rotation

$$V = \pi \int_a^b R^2 \, dx$$

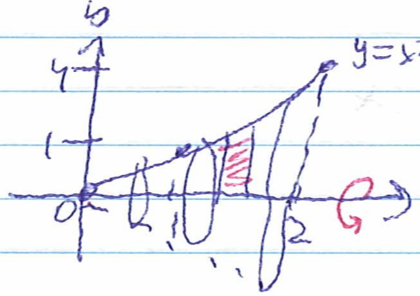
Vertical Axis of Rotation

$$V = \pi \int_c^d R^2 \, dy$$

Ex.1 Revolve $y=x^2$ over $[0,2]$ about x-axis
from disk method:



| $x$ | $y=x^2$ |
| --- | --- |
| 0 | 0 |
| 1 | 1 |
| 2 | 4 |



Disk method (horizontal axis of rotation)

$$V=\int_0^2 \pi (x^2)^2 dx$$

$$=\int_0^2 \pi x^4 dx$$

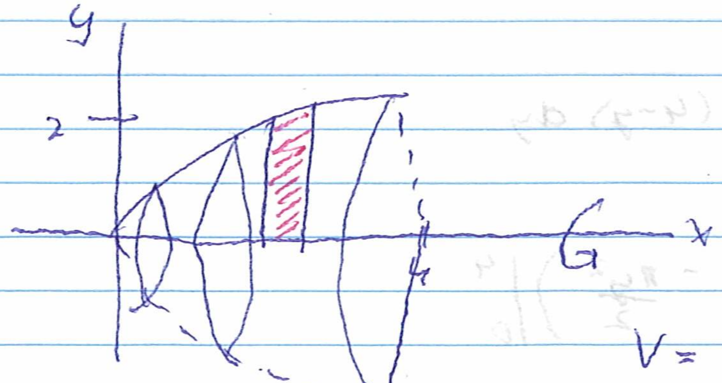
Radius R: $R=x^2-0=x^2$

$$=\frac{\pi}{5}x^5\Big|_0^2$$

$$=\frac{32\pi}{5} \text{ units}^3$$

Ex.2 Find volume of solid formed by revolving $y=\sqrt{x}$ over $[1,4]$ about x-axis.

| $x$ | $y=\sqrt{x}$ |
| --- | --- |
| 0 | 0 |
| 1 | 1 |
| 4 | 2 |



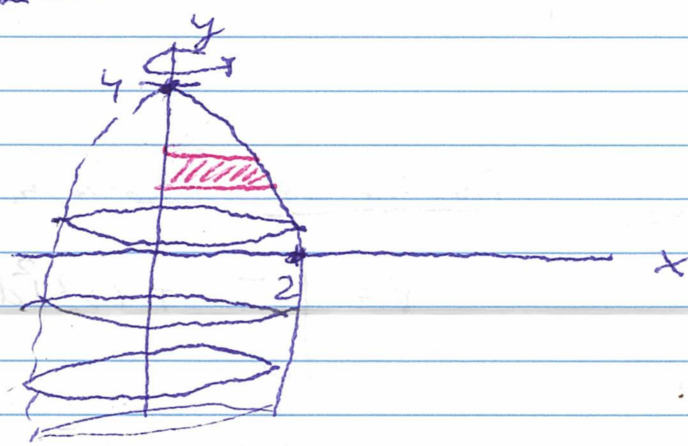
Disk method

$$V=\int_1^4 \pi (\sqrt{x})^2 dx$$

$$=\int_1^4 \pi x\, dx = \pi\frac{x^2}{2}\Big|_1^4$$

$= \pi\left(\frac{16}{2}\right) - \pi\left(\frac{1}{2}\right) = \boxed{\frac{15\pi}{2}}$

Ex-3. Find the volume of the solid formed by revolving $x = \sqrt{4-y}$ over $[0,4]$ about y-axis.




| $y$ | $x = \sqrt{4-y}$ | | |
|---|---|---|---|
| 0 | $x = \sqrt{4} = 2$ | | $(2,0)$ |
| 4 | $x = 0$ | $(0,4)$ | |

Disk method : Vertical axis of rotation

$$\text{Volume} = \int_0^4 \pi\left(\sqrt{4-y}\right)^2 dy$$

$$= \pi\int_0^4 (4-y)\, dy$$

$$= \left(4\pi y - \frac{\pi y^2}{2}\right)\Big|_0^4$$

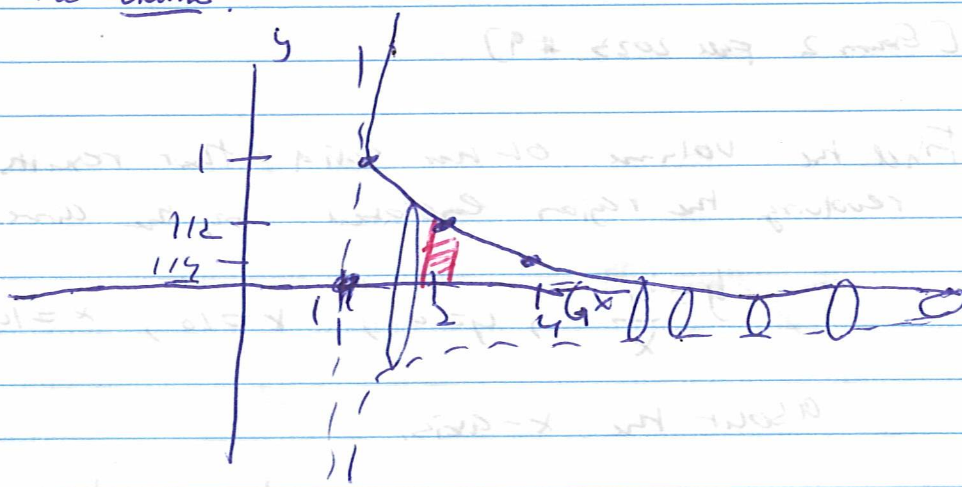
$$= 16\pi - 8\pi = \boxed{8\pi}$$

Ex 4 [Gabriel's horn]

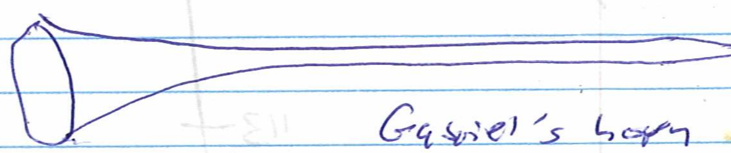
Consider the surface area and volume of the solid formed by rotating region bounded by

x-axis, $x=1$, and $y=\frac{1}{x}$ around the x-axis.

Find the volume.




| $x$ | $y=\frac{1}{x}$ |
|---|---|
| 1 | 1 |
| 2 | 1/2 |
| 4 | 1/4 |



Gabriel's horn

Radius: $\frac{1}{x} - 0 = \frac{1}{x}$

Disk method: Volume $= \int_1^\infty \pi \left(\frac{1}{x}\right)^2 dx$

$= \int_1^\infty \pi \frac{1}{x^2}\, dx$

$= \pi \int_1^\infty \frac{1}{x^2}\, dx$

$\lim_{a\to\infty} \pi \int_1^a \frac{1}{x^2}\, dx = \pi\left(1-\frac{1}{a}\right) \to \boxed{\pi}$ as $a \to \infty$.

Volume: $\pi$ units$^3$

Fact: Surface area $= \boxed{\infty}$

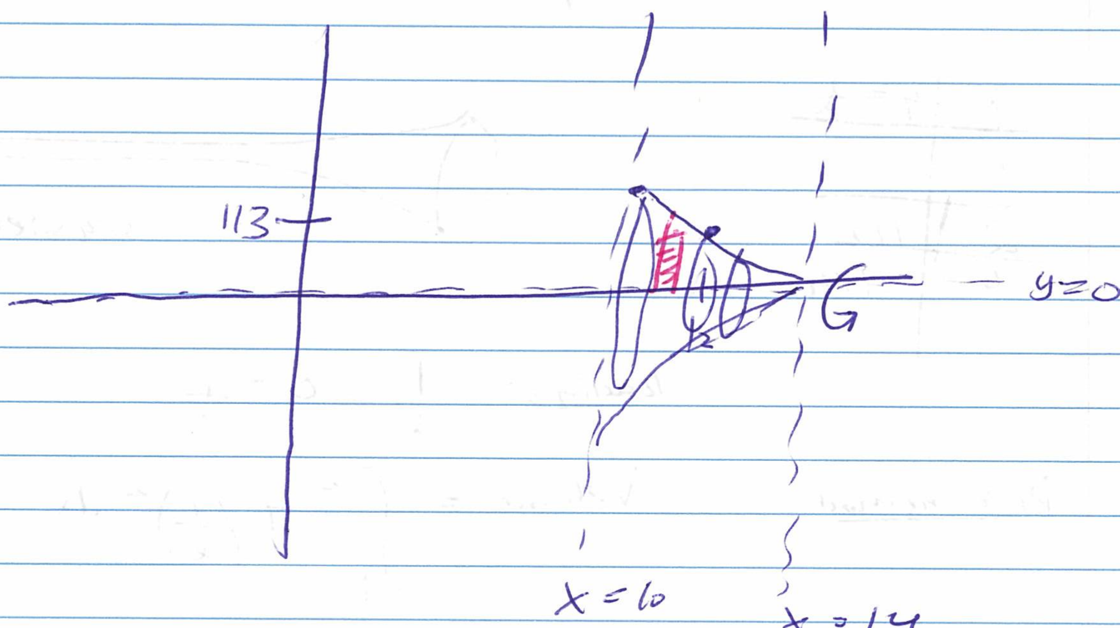
We won't learn in this class surface area of solids of revolution (MA 162/166 covers this)

Ex 45 [Exam 2 Fall 2023 #9]

Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = \frac{4}{x}, \quad y = 0, \quad x = 10, \quad x = 14$$

about the x-axis.




| $x$ | $y = 4/x$ |
|---|---|
| 12 | $4/12 = 1/3$ |
| 10 | $4/10 = 2/5$ |

Disk.

$$\text{Volume} = \pi \int_{10}^{14} \left(\frac{4}{x}\right)^2 dx$$

$$= \pi \int_{10}^{14} \frac{16}{x^2}\, dx$$

$$= \pi\left[\frac{-16}{x}\right]_{10}^{14} = \pi\left(\frac{-16}{14} + \frac{16}{10}\right)$$

$$= 16\pi\left(\frac{-1}{14} + \frac{1}{10}\right)$$

$$= 16\pi\left(\frac{-5}{70} + \frac{7}{70}\right)$$

$$= 16\pi\left(\frac{2}{70}\right) = \frac{32\pi}{70} = \frac{16\pi}{35}$$

$$\boxed{\text{Volume} = \frac{16\pi}{35}}$$

$\boxed{\text{in exam: D}}$