MA 16020 – Applied Calculus II: Lecture 11
Improper Integrals

Warm-Up: Limits Refresher

Notation:

$\lim_{x \to a} f(x)$ means the value $f(x)$ approaches as x gets close to a.

Basic examples

- $\lim_{x \to 2} (3x + 1) = 7$
- $\lim_{x \to 0} \frac{\sin x}{x} = 1$
- $\lim_{x \to \infty} \frac{1}{x} = 0$
- $\lim_{x \to 0^+} \frac{1}{x} = +\infty$

Warm-Up: Limits Refresher

Notation:

$\lim_{x \to a} f(x)$ means the value $f(x)$ approaches as x gets close to a.

Basic examples

- $\lim_{x \to 2} (3x + 1) = 7$
- $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$
- $\lim_{x \to \infty} \dfrac{1}{x} = 0$
- $\lim_{x \to 0^+} \dfrac{1}{x} = +\infty$

Idea: Improper integrals use limits to "patch" situations where the Fundamental Theorem of Calculus does not apply.

Fundamental Theorem of Calculus

Reminder

If f is **continuous** on the closed interval $[a, b]$ and F is an antiderivative of f, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Key conditions (must hold):

- f is continuous on $[a, b]$
- Both endpoints a, b are finite

Question: What if one or both condiions fail?

A Trap: $\int_{-1}^{1} \frac{1}{x^2}\,dx$

Naive computation (treating as ordinary integral):

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_{-1}^{1} = \Big(-1\Big) - \Big(1\Big) = -2.$$

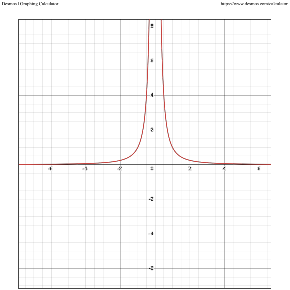
A Trap: $\int_{-1}^{1} \frac{1}{x^2}\,dx$

Naive computation (treating as ordinary integral):

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_{-1}^{1} = \Big(-1\Big) - \Big(1\Big) = -2.$$

The trap: This answer is nonsense.

- $f(x) = \frac{1}{x^2}$ is **not continuous** on the closed interval $[-1, 1]$.
- There is a vertical asymptote at $x = 0$.
- The Fundamental Theorem of Calculus *cannot be applied*.

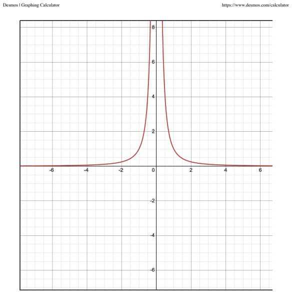


What Went Wrong with $\int_{-1}^{1} \frac{1}{x^2}\, dx$?

- The shaded area near $x = 0$ **grows without bound**.
- $f(x) = \frac{1}{x^2}$ is **discontinuous at** $x = 0$.
- Therefore, the **Fundamental Theorem of Calculus fails** on $[-1, 1]$.

Conclusion

This is an example of an **improper integral of Type II** (finite interval, but $f(x)$ has an infinite discontinuity).



Sometimes we want to integrate functions outside the scope of the FTC:

- **Infinite intervals:** $[a, \infty)$, $(-\infty, b]$, or $(-\infty, \infty)$.
- **Infinite discontinuities:** $f(x)$ has a vertical asymptote inside $[a, b]$.

Two types of improper integrals

1. **Type I:** Infinite interval.
2. **Type II:** Infinite discontinuity of $f(x)$ on a finite interval.

Definition: Improper Integrals of Type I

Case 1: Infinite upper bound

$$\int_a^\infty f(x)\,dx := \lim_{t\to\infty} \int_a^t f(x)\,dx$$

Case 2: Infinite lower bound

$$\int_{-\infty}^b f(x)\,dx := \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

Case 3: Both sides infinite

$$\int_{-\infty}^\infty f(x)\,dx := \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

Definition: Improper Integrals of Type I

Case 1: Infinite upper bound

$$\int_a^{\infty} f(x)\,dx := \lim_{t\to\infty} \int_a^t f(x)\,dx$$

Case 2: Infinite lower bound

$$\int_{-\infty}^{b} f(x)\,dx := \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

Case 3: Both sides infinite

$$\int_{-\infty}^{\infty} f(x)\,dx := \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx$$

Convergence: These integrals only make sense if the relevant limits *exist and are finite*.

Example 1: Type I Improper Integrals

Determine whether each integral converges (has a limit) or diverges (is either ∞ or DNE).

(a) $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$

(b) $\displaystyle\int_0^\infty 5xe^{-x}\,dx$

(c) $\displaystyle\int_2^\infty \frac{dx}{x}$

(d) $\displaystyle\int_{-\infty}^\infty x\sin(x^2)\,dx$

Definition: Improper Integrals of Type II

Suppose f has a vertical asymptote (infinite discontinuity) at one endpoint of the interval.

Right-endpoint singularity

If f is continuous on $[a, b)$ and discontinuous at b, then

$$\int_a^b f(x)\,dx := \lim_{t\to b^-} \int_a^t f(x)\,dx$$

provided the limit exists (finite).

Left-endpoint singularity

If f is continuous on $(a, b]$ and discontinuous at a, then

$$\int_a^b f(x)\,dx := \lim_{t\to a^+} \int_t^b f(x)\,dx$$

provided the limit exists (finite).

Example 2: Type II Improper Integrals

Determine whether each integral converges or diverges.

(a) $\displaystyle\int_0^{\frac{\pi}{2}} \tan x \, dx$

(b) $\displaystyle\int_{-2}^{14} \frac{1}{\sqrt[4]{x+2}} \, dx$

Solution to Example 2(a)

$$\int_0^{\frac{\pi}{2}} \tan x \, dx = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\cos x} \, dx$$

Let $u = \cos x$, so $du = -\sin x \, dx$:

$$\int_0^{\frac{\pi}{2}} \tan x \, dx = -\int_1^0 \frac{1}{u} \, du = \int_0^1 \frac{1}{u} \, du$$

But

$$\int_0^1 \frac{1}{u} \, du = \lim_{t \to 0^+} \int_t^1 \frac{1}{u} \, du = \lim_{t \to 0^+} \Big[\ln u \Big]_t^1 = \lim_{t \to 0^+} (0 - \ln t).$$

$$= +\infty$$

Conclusion

$$\int_0^{\pi/2} \tan x \, dx \text{ diverges.}$$

Solution to Example 2(b)

$$\int_{-2}^{14} \frac{1}{\sqrt[4]{x+2}}\,dx = \lim_{t\to -2^+} \int_t^{14} (x+2)^{-1/4}\,dx$$

Antiderivative:

$$\int (x+2)^{-1/4}\,dx = \frac{(x+2)^{3/4}}{3/4} + C = \tfrac{4}{3}(x+2)^{3/4}.$$

Evaluate:

$$\lim_{t\to -2^+} \left[\tfrac{4}{3}(x+2)^{3/4}\right]_t^{14} = \tfrac{4}{3}\Big((16)^{3/4} - \lim_{t\to -2^+}(t+2)^{3/4}\Big).$$

Since $(t+2)^{3/4} \to 0$ as $t \to -2^+$:

$$= \tfrac{4}{3}\cdot(16^{3/4}) = \tfrac{4}{3}\cdot 8 = \tfrac{32}{3}.$$

Conclusion

$\displaystyle\int_{-2}^{14} \frac{1}{\sqrt[4]{x+2}}\,dx$ converges and equals $\frac{32}{3}$.