

MA 16020 – Applied Calculus II: Lecture 10

Integration by Partial Fractions II

Type 2: Repeated Linear Factors

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When the denominator contains powers of the same linear factor, e.g. $(x - a)^n$, the partial fraction decomposition requires a term for *each power*:

$$\frac{P(x)}{(x - a)^n} = \frac{A_1}{x - a} + \frac{A_2}{(x - a)^2} + \cdots + \frac{A_n}{(x - a)^n}.$$

Key points:

- Each A_i is a constant to solve.
- Multiply through by the denominator and either:
 - Plug in convenient values of x (especially $x = a$ for the highest power term), or
 - Match coefficients.

Today we will do simple examples to illustrate.

Type 2 Partial Fraction Examples

Find the partial fraction decomposition of the following Type 2 rational functions and evaluate the integrals:

1 $\frac{3x + 5}{x^2 - 6x + 9}$ (Hint: factor denominator if possible)

2 $\frac{3x + 7}{(x - 2)^2}$

Steps:

- Factor the denominator (if repeated linear factor, write separate terms for each power)
- Set up partial fraction decomposition
- Solve for constants
- Integrate each term

(Constants and integration to be done on board)

Type 3: Irreducible Quadratic Factors

Type 3: Distinct Irreducible Quadratic Factors

If the denominator has an irreducible quadratic factor (cannot be factored over the reals), e.g. $(x^2 + bx + c)$, the numerator must be a **linear expression**:

$$\frac{P(x)}{(x - a)(x^2 + bx + c)} = \frac{A}{x - a} + \frac{Bx + C}{x^2 + bx + c}.$$

Example:

$$\frac{2x + 3}{(x - 1)(x^2 + 1)} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + 1}.$$

Key idea: For each quadratic factor, use $(Bx + C)$ in the numerator (never just a constant).

Type 3 Example 1

Decompose into partial fractions:

$$\frac{3x^2 + 7x + 28}{x(x^2 + x + 7)}$$

Step 1: Set up the form

$$\frac{3x^2 + 7x + 28}{x(x^2 + x + 7)} = \frac{A}{x} + \frac{Bx + C}{x^2 + x + 7}$$

Step 2: Multiply through by denominator

$$\frac{3x^2 + 7x + 28}{x(x^2 + x + 7)} = \frac{A(x^2 + x + 7) + Bx + C(x)}{x(x^2 + x + 7)}$$

Step 3: Solve for constants (done on board).

Type 4: Repeated Irreducible Quadratic Factors

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If the denominator has a repeated irreducible quadratic factor $(x^2 + bx + c)^n$, we need a linear numerator for *each power*:

$$\frac{P(x)}{(x^2 + bx + c)^n} = \frac{B_1x + C_1}{x^2 + bx + c} + \frac{B_2x + C_2}{(x^2 + bx + c)^2} + \cdots + \frac{B_nx + C_n}{(x^2 + bx + c)^n}.$$

Example:

$$\frac{x^2 + 5}{(x^2 + 1)^2} = \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2}.$$

Key idea: Each quadratic factor requires a linear numerator at each power.

Type 4 Partial Fraction Example

Decompose the following rational function:

$$\frac{4x^3 + 16x + 7}{(x^2 + 4)^2}.$$

Step 1: Set up the form (Type 4: repeated irreducible quadratic)

$$\frac{4x^3 + 16x + 7}{(x^2 + 4)^2} = \frac{Ax + B}{x^2 + 4} + \frac{Cx + D}{(x^2 + 4)^2}.$$

Step 2: Multiply both sides by the denominator

$$4x^3 + 16x + 7 = (Ax + B)(x^2 + 4) + (Cx + D).$$

Step 3: Expand the first term

$$(Ax + B)(x^2 + 4) = Ax^3 + 4Ax + Bx^2 + 4B.$$

Step 4: Combine with second term

$$4x^3 + 16x + 7 = Ax^3 + Bx^2 + 4Ax + 4B + Cx + D.$$

Identify the Correct Partial Fraction Decomposition

Consider the rational function:

$$f(x) = \frac{32x + 5}{(x^2 - 36)(x^2 + 25)}.$$

Which of the following could represent a valid partial fraction decomposition?

- A. $\frac{Ax-6}{x-6} + \frac{Bx+6}{x+6} + \frac{Cx+D}{x^2+25}$
- B. $\frac{A}{x-6} + \frac{B}{x+6} + \frac{Cx+D}{x^2+25}$
- C. $\frac{Ax+B}{x-6} + \frac{Cx-5}{x+6} + \frac{Dx+5}{x^2+25}$
- D. $\frac{Ax-6}{x-6} + \frac{Bx+6}{x+6} + \frac{Cx-5}{x^2+25} + \frac{Dx+5}{x^2+25}$

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Correct answer: B. Each distinct linear factor gets a constant numerator and each irreducible quadratic gets a linear numerator.

Identify the Correct Partial Fraction Decomposition

Consider the rational function:

$$f(x) = \frac{6}{(x^2 - 100)^2} = \frac{6}{(x - 10)^2(x + 10)^2}.$$

Which of the following could represent a valid partial fraction decomposition?

- A. $\frac{A}{x-10} + \frac{B}{(x-10)^2} + \frac{C}{x+10} + \frac{D}{(x+10)^2}$
- B. $\frac{Ax+B}{x^2-100} + \frac{Cx+D}{(x^2-100)^2}$
- C. $\frac{A}{(x^2-100)^2} + \frac{B}{x^2-100}$
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- C. $\frac{A}{(x^2-100)^2} + \frac{B}{x^2-100}$
- D. $\frac{A}{x-10} + \frac{B}{x+10}$

Correct answer: A. After factoring the quadratic, each repeated linear factor gets a term for each power.

Identify the Correct Partial Fraction Decomposition

Consider the rational function:

$$f(x) = \frac{15}{8x^3 - 9x^2}.$$

Factor the denominator:

$$8x^3 - 9x^2 = x^2(8x - 9).$$

Which of the following could represent a valid partial fraction decomposition?

- A. $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{8x-9}$
- B. $\frac{A}{x} + \frac{B}{8x-9}$
- C. $\frac{Ax+B}{x^2} + \frac{C}{8x-9}$
- D. $\frac{Ax+B}{x} + \frac{Cx+D}{8x-9}$

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- D. $\frac{Ax+B}{x} + \frac{Cx+D}{8x-9}$

Correct answer: A. Each repeated linear factor (x^2) gets a term for each power, and the distinct linear factor ($8x - 9$) gets a constant numerator.