MA 16020 Lesson 33: Additional Notes

In this lesson we continue computing double integrals but with variable limits. Recall that in one-variable calculus computing a definite integral results in a number. But as we've seen, computing the inside integral produces a function. So there's really nothing special about the limits of integration of the inside integral being numbers. When integrating, note: the variable in the bounds of the *outer* integral is treated as a constant with respect to the *inner* variable.

Example 1

Compute

$$\int_5^6 \int_0^y 8xy\,dx\,dy.$$

Solution.

$$\int_5^6 \left(\int_0^y 8xy\,dx \right) dy.$$

Treat y as a constant when integrating with respect to x:

$$\int_0^y 8xy\,dx = 8y \int_0^y x\,dx = 8y \left[\frac{x^2}{2} \right]_0^y = 8y \cdot \frac{y^2}{2} = 4y^3.$$

Now integrate with respect to y:

$$\int_5^6 4y^3\,dy = 4 \left[\frac{y^4}{4} \right]_5^6 = \left[y^4 \right]_5^6 = 6^4 - 5^4 = 1296 - 625 = 671.$$

$$\boxed{671}$$

Example 2

Compute

$$\int_0^{\sqrt{\pi/2}} \int_0^{x^2} -4x \cos y \, dy \, dx.$$

Solution.
Evaluate the inner integral with respect to y:

$$\int_0^{x^2} -4x \cos y \, dy = -4x \left[\sin y\right]_0^{x^2} = -4x \sin(x^2).$$

Now integrate with respect to x:

$$\int_0^{\sqrt{\pi/2}} -4x \sin(x^2) \, dx.$$

Use substitution $u = x^2$, so $du = 2x\,dx$, $x\,dx = \frac{1}{2}du$. When $x = 0$, $u = 0$. When $x = \sqrt{\pi/2}$, $u = \pi/2$.

$$-4 \int_0^{\sqrt{\pi/2}} x \sin(x^2) \, dx = -4 \int_0^{\pi/2} \sin(u) \, \frac{1}{2} \, du = -2 \int_0^{\pi/2} \sin(u) \, du.$$

$$-2 \left[-\cos u\right]_0^{\pi/2} = 2 \left(\cos\left(\frac{\pi}{2}\right) - \cos(0)\right) = 2(0 - 1) = -2.$$

$$\boxed{-2}$$