Shell Method Solutions: Lesson 17-18

Example 1

Problem: Find the volume obtained by revolving the region bounded by

$$y = 2x^2 - x^3 \quad \text{and} \quad y = 0$$

about the y-axis.

Solution:

- The region lies between $x = 0$ and $x = 2$ (where $y = 2x^2 - x^3 = 0$).
- Using the shell method, radius $r = x$, height $h = 2x^2 - x^3$.

$$V = 2\pi \int_0^2 rh\,dx = 2\pi \int_0^2 x(2x^2 - x^3)\,dx = 2\pi \int_0^2 (2x^3 - x^4)\,dx$$

$$V = 2\pi \left[\frac{2x^4}{4} - \frac{x^5}{5}\right]_0^2 = 2\pi \left(8 - \frac{32}{5}\right) = 2\pi \left(\frac{8}{5}\right) = \frac{16\pi}{5}.$$

$$\boxed{V = \frac{16\pi}{5}}$$

Example 2

Problem: Find the volume obtained by revolving the region bounded by

$$y = \frac{5}{x^3}, \quad y = 0, \quad x = 1, \quad x = 5$$

about the y-axis.

Solution:

- The radius is $r = x$.
- The height is $h = \dfrac{5}{x^3}$.

$$V = 2\pi \int_1^5 rh\,dx = 2\pi \int_1^5 x\left(\frac{5}{x^3}\right)dx = 10\pi \int_1^5 x^{-2}\,dx$$

$$V = 10\pi\left[-x^{-1}\right]_1^5 = 10\pi\left(-\frac{1}{5}+1\right) = 10\pi\left(\frac{4}{5}\right) = 8\pi.$$

$$\boxed{V = 8\pi}$$

Example 3

Problem: Find the volume obtained by revolving the region bounded by

$$x = y^2 - 2y \quad \text{and} \quad x = 4y - y^2$$

about the x-axis.

Solution:

- Intersections: $y^2 - 2y = 4y - y^2 \implies 2y^2 - 6y = 0 \implies y = 0, 3.$
- The radius is $r = y$, the height is

$$h = (4y - y^2) - (y^2 - 2y) = 6y - 2y^2.$$

$$V = 2\pi \int_0^3 rh\,dy = 2\pi \int_0^3 y(6y - 2y^2)\,dy = 2\pi \int_0^3 (6y^2 - 2y^3)\,dy$$

$$V = 2\pi\left[2y^3 - \frac{y^4}{2}\right]_0^3 = 2\pi\,(54 - 40.5) = 2\pi(13.5) = 27\pi.$$

$$\boxed{V = 27\pi}$$

Example 4 (Set-Up and Evaluation)

Problem: Set up the integral for the volume of the region bounded by

$$y = 4x, \quad y = 0, \quad x = 10$$

when the region is revolved about the **y-axis**.

(a) Disk/Washer method:

For a fixed y (with $0 \le y \le 40$) the horizontal slice runs from $x_{\text{left}} = y/4$ to $x_{\text{right}} = 10$, so

$$V = \pi \int_0^{40} \left(10^2 - \left(\frac{y}{4}\right)^2\right) dy = \pi \int_0^{40} \left(100 - \frac{y^2}{16}\right) dy = \frac{8000\pi}{3}.$$

(b) Shell method:
Vertical shells at $x \in [0, 10]$ have radius $r = x$ and height $h = 4x$, so

$$V = 2\pi \int_0^{10} x(4x)\, dx = 8\pi \int_0^{10} x^2\, dx = \frac{8000\pi}{3}.$$

Lesson 18 – Shell Method: Revolving Around Lines

Problem: Using the **Shell Method**, set up and evaluate the integral that represents the volume of the solid obtained by revolving the region bounded by the triangle with vertices

$$(0,0),\ (2,0),\ (2,3)$$

about the given lines.

Step 1. Equation of the slanted side
The slanted side passes through $(0,0)$ and $(2,3)$, so its slope is

$$m = \frac{3-0}{2-0} = \frac{3}{2}.$$

Hence, the equation of the line is

$$y = \frac{3}{2}x.$$

(a) Revolve about $x = 3$

Using the Shell Method:

$$V = 2\pi \int_0^2 (\text{radius})(\text{height})\, dx.$$

Radius: $r(x) = 3 - x$
Height: $h(x) = \frac{3}{2}x$

$$\begin{aligned}
V &= 2\pi \int_0^2 (3-x)\left(\frac{3}{2}x\right) dx \\
&= 3\pi \int_0^2 x(3-x)\, dx \\
&= 3\pi \int_0^2 (3x - x^2)\, dx \\
&= 3\pi \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^2 \\
&= 3\pi \left(6 - \frac{8}{3}\right) \\
&= 3\pi \left(\frac{10}{3}\right) = 10\pi.
\end{aligned}$$

$$\boxed{V = 10\pi}$$

(b) Revolve about $x = -1$

Radius: $r(x) = x - (-1) = x + 1$
Height: $h(x) = \frac{3}{2}x$

$$\begin{aligned}
V &= 2\pi \int_0^2 (x+1)\left(\frac{3}{2}x\right) dx \\
&= 3\pi \int_0^2 (x^2 + x)\, dx \\
&= 3\pi \left[\frac{x^3}{3} + \frac{x^2}{2}\right]_0^2 \\
&= 3\pi \left(\frac{8}{3} + 2\right) \\
&= 3\pi \left(\frac{14}{3}\right) = 14\pi.
\end{aligned}$$

$$\boxed{V = 14\pi}$$

0.1 Example 2

Problem 2: Using the **Shell Method**, find the volume of the solid obtained by revolving the region bounded by the triangle with vertices

$$(0,0),\ (5,0),\ (5,6)$$

about the horizontal line $\mathbf{y = 6}$.

Step 1. Equation of the slanted side

The slanted side passes through $(0,0)$ and $(5,6)$, so its slope is

$$m = \frac{6-0}{5-0} = \frac{6}{5}.$$

Hence, the equation of the line is

$$y = \frac{6}{5}x.$$

Step 2. Shell Method Setup

- Revolving about a horizontal line $y = 6 \rightarrow$ use **horizontal shells**, integrate w.r.t. y.
- For a typical shell at height y:

$$\text{Radius: } r(y) = 6 - y$$

$$\text{Height: } h(y) = x_{\text{right}} - x_{\text{left}} = 5 - \frac{5}{6}y$$

- Thickness: dy, $y \in [0,6]$.

$$V = 2\pi \int_0^6 (6-y)\left(5 - \frac{5}{6}y\right) dy$$

Step 3. Simplify the integrand

$$(6-y)\left(5 - \frac{5}{6}y\right) = \frac{5}{6}y^2 - 10y + 30$$

Step 4. Integrate and evaluate

$$\int_0^6 \left(\frac{5}{6}y^2 - 10y + 30\right) dy = \frac{5}{18}y^3 - 5y^2 + 30y\Big|_0^6 = 60$$

$$V = 2\pi \cdot 60 = \boxed{120\pi}$$

Summary:

Axis of Revolution	**Method**	**Volume**
$y = 6$	Shell	120π

Lesson 18 — Example 3

Problem: Using the **Shell Method**, find the volume of the solid obtained by revolving the region bounded by

$$y = \sqrt{x}, \quad y = 0, \quad x = 4$$

about the vertical line $\mathbf{x = 5}$.

Step 1. Shell Method Setup

- Revolving about $x = 5$ (vertical line) $\Rightarrow$ use **vertical shells**, integrate w.r.t. x. - Radius: $r(x) = 5 - x$ - Height: $h(x) = \sqrt{x} - 0 = \sqrt{x}$ - Thickness: dx, $x \in [0, 4]$

$$V = 2\pi \int_0^4 (5 - x)\sqrt{x}\, dx$$

Step 2. Simplify and integrate

$$V = 2\pi \int_0^4 \left(5x^{1/2} - x^{3/2}\right) dx = 2\pi \left[\frac{10}{3}x^{3/2} - \frac{2}{5}x^{5/2}\right]_0^4 = 2\pi \left(\frac{80}{3} - \frac{64}{5}\right) = 2\pi \cdot \frac{208}{15} = \frac{416\pi}{15}$$

$$\boxed{V = \frac{416\pi}{15}}$$

Summary:

Axis of Revolution	Method	Volume
$x = 5$	Shell	$\frac{416\pi}{15}$

Example 3(b)

Problem: Using the **Shell Method**, find the volume of the solid obtained by revolving the region bounded by

$$y = \sqrt{x}, \quad y = 0, \quad x = 4$$

about the vertical line $\mathbf{x = -1}$.

Step 1. Shell Method Setup

- Revolving about $x = -1$ (vertical line) $\Rightarrow$ use **vertical shells**, integrate w.r.t. x. - Radius: $r(x) = x - (-1) = x + 1$ - Height: $h(x) = \sqrt{x}$ - Thickness: dx, $x \in [0, 4]$

$$V = 2\pi \int_0^4 (x + 1)\sqrt{x}\, dx = 2\pi \int_0^4 (x^{3/2} + x^{1/2})\, dx$$

Step 2. Integrate and evaluate

$$\int_0^4 x^{3/2}dx = \frac{64}{5}, \quad \int_0^4 x^{1/2}dx = \frac{16}{3}$$

$$V = 2\pi\left(\frac{64}{5} + \frac{16}{3}\right) = 2\pi \cdot \frac{272}{15} = \frac{544\pi}{15}$$

$$\boxed{V = \frac{544\pi}{15}}$$

Summary:

Axis of Revolution	Method	Volume
$x = -1$	Shell	$\frac{544\pi}{15}$

Lesson 18 — Example 4

Problem: Using the **Shell Method**, find the volume of the solid obtained by revolving the region bounded by

$$x = y^2 + 1, \quad x = 2$$

about the horizontal lines:

(a) $y = 3$

(b) $y = -2$

(a) Revolve about $y = 3$

- Horizontal shells $\rightarrow$ integrate w.r.t y, $y \in [-1, 1]$ - Radius: $r(y) = 3 - y$ - Height: $h(y) = 2 - (y^2 + 1) = 1 - y^2$

$$V = 2\pi \int_{-1}^{1} (3 - y)(1 - y^2)\, dy = 2\pi \int_{-1}^{1} (y^3 - 3y^2 - y + 3)\, dy$$

$$\int_{-1}^{1} (y^3 - 3y^2 - y + 3)\, dy = 4 \quad \Rightarrow \quad V = 2\pi \cdot 4 = \boxed{8\pi}$$

(b) Revolve about $y = -2$

- Radius: $r(y) = y + 2$ - Height: $h(y) = 1 - y^2$

$$V = 2\pi \int_{-1}^{1} (y+2)(1-y^2)\, dy = 2\pi \int_{-1}^{1} (-y^3 - 2y^2 + y + 2)\, dy$$

$$\int_{-1}^{1} (-y^3 - 2y^2 + y + 2)\, dy = \frac{7}{3} \quad \Rightarrow \quad V = 2\pi \cdot \frac{7}{3} = \boxed{\frac{14\pi}{3}}$$

Summary:

Axis of Revolution	**Method**	**Volume**
$y = 3$	Shell	8π
$y = -2$	Shell	$\frac{14\pi}{3}$