MA 16020 — Final Exam Guide Selected Solutions

Problem 1

Evaluate the integral

$$\int \frac{1}{(3x-1)^4}\,dx.$$

Solution 1: U-sub

We can use a u-substitution. Let

$$u = 3x - 1 \implies du = 3\,dx \implies dx = \frac{du}{3}.$$

Then the integral becomes:

$$\int \frac{1}{(3x-1)^4}\,dx = \int \frac{1}{u^4} \cdot \frac{du}{3} = \frac{1}{3}\int u^{-4}\,du.$$

Integrate:

$$\frac{1}{3} \cdot \frac{u^{-3}}{-3} = -\frac{1}{9}u^{-3} = -\frac{1}{9(3x-1)^3}.$$

Include the constant of integration:

$$\boxed{-\frac{1}{9(3x-1)^3} + C}$$

Problem 2

Evaluate

$$\int e^{3-2x}\,dx.$$

Solution 2

We can use a u-substitution. Let

$$u = 3 - 2x \implies du = -2\,dx \implies dx = -\frac{1}{2}du.$$

Then the integral becomes:

$$\int e^{3-2x}\,dx = \int e^u \left(-\frac{1}{2}du\right) = -\frac{1}{2}\int e^u\,du.$$

Integrate:

$$-\frac{1}{2}e^u + C.$$

Back-substitute $u = 3 - 2x$:

$$\boxed{-\frac{1}{2}e^{3-2x} + C}.$$

Problem 3: Tangent Line Slope

Find a function $f(x)$ whose tangent line has slope

$$f'(x) = x\sqrt{5 - x^2}$$

and whose graph passes through the point $(2, 10)$.

Solution 3

We are given

$$f'(x) = x\sqrt{5 - x^2}.$$

Integrate both sides:

$$f(x) = \int x\sqrt{5 - x^2}\, dx + C.$$

Use the substitution $u = 5 - x^2$, so that $du = -2x\, dx \implies x\, dx = -\frac{1}{2}du$:

$$\int x\sqrt{5 - x^2}\, dx = -\frac{1}{2}\int u^{1/2}\, du = -\frac{1}{3}u^{3/2} + C.$$

Back-substitute $u = 5 - x^2$:

$$f(x) = -\frac{1}{3}(5 - x^2)^{3/2} + C.$$

Apply the initial condition $f(2) = 10$:

$$10 = -\frac{1}{3}(5 - 2^2)^{3/2} + C = -\frac{1}{3} + C \implies C = \frac{31}{3}.$$

$$\boxed{f(x) = -\frac{1}{3}(5 - x^2)^{3/2} + \frac{31}{3}}$$

This function satisfies both the derivative condition and the point $(2, 10)$.

Problem 4

Evaluate

$$\int x \ln\left(x^2\right) dx$$

Solution 4

Notice that $\ln(x^2) = 2\ln(x)$. So the integral becomes:

$$\int x\ln(x^2)\,dx = \int x \cdot 2\ln(x)\,dx = 2\int x\ln(x)\,dx.$$

Use **integration by parts**. Let

$$u = \ln(x) \implies du = \frac{1}{x}dx, \quad dv = x\,dx \implies v = \frac{x^2}{2}.$$

Then

$$\int x\ln(x)\,dx = uv - \int v\,du = \frac{x^2}{2}\ln(x) - \int \frac{x^2}{2}\cdot\frac{1}{x}dx = \frac{x^2}{2}\ln(x) - \int \frac{x}{2}dx.$$

Compute the remaining integral:

$$\int \frac{x}{2}dx = \frac{1}{2}\cdot\frac{x^2}{2} = \frac{x^2}{4}.$$

So

$$\int x\ln(x)\,dx = \frac{x^2}{2}\ln(x) - \frac{x^2}{4}.$$

Multiply by 2 (from the substitution $\ln(x^2) = 2\ln(x)$):

$$\int x\ln(x^2)\,dx = 2\left(\frac{x^2}{2}\ln(x) - \frac{x^2}{4}\right) = x^2\ln(x) - \frac{x^2}{2}.$$

Include the constant of integration:

$$\boxed{x^2\ln(x) - \frac{x^2}{2} + C}.$$

Problem 5

The area of the region bounded by the curves $y = x^2 + 1$ and $y = 3x + 5$ is

Solution 5

Find the points of intersection:

$$x^2 + 1 = 3x + 5 \implies x^2 - 3x - 4 = 0 \implies x = -1, 4.$$

Determine which function is on top: test $x = 0$, $y_{\text{line}} = 5$, $y_{\text{parabola}} = 1$, so the line is above.
Set up the area integral:

$$A = \int_{-1}^{4} \big[(3x + 5) - (x^2 + 1)\big]\, dx = \int_{-1}^{4} (-x^2 + 3x + 4)\, dx.$$

Integrate:

$$\int (-x^2 + 3x + 4) dx = -\frac{x^3}{3} + \frac{3x^2}{2} + 4x.$$

Evaluate at the bounds:

$$F(4) = -\frac{64}{3} + 24 + 16 = \frac{56}{3}, \quad F(-1) = \frac{1}{3} + \frac{3}{2} - 4 = -\frac{13}{6}.$$

Subtract:

$$A = F(4) - F(-1) = \frac{56}{3} - \left(-\frac{13}{6}\right) = \frac{56}{3} + \frac{13}{6} = \frac{125}{6}.$$

$$\boxed{\frac{125}{6}}$$

Problem 6

If $f(x, y) = (xy + 1)^2 - \sqrt{y^2 - x^2}$, evaluate $f(-2, 1)$

Solution 6

Substitute $x = -2$ and $y = 1$ into the function:

$$f(-2, 1) = ((-2)(1) + 1)^2 - \sqrt{1^2 - (-2)^2}.$$

Compute each part:
1. The first term:

$$(-2 \cdot 1 + 1)^2 = (-2 + 1)^2 = (-1)^2 = 1.$$

2. The second term:

$$\sqrt{1^2 - (-2)^2} = \sqrt{1 - 4} = \sqrt{-3}.$$

Since $\sqrt{-3} = i\sqrt{3}$ (complex number), we have the function is **undefined**.

Problem 7

A paint store carries two brands of latex paint. Sales figures indicate that if the first brand is sold for x_1 dollars per gallon and the second for x_2 dollars per gallon, and the demand for the first brand will be $D(x_1, x_2) = 100 + 5x_1 - 10x_2$ gallons per month and the second brand will be $D_2(x_1, x_2) = 200 - 10x_1 + 15x_2$ gallons per month. Express the paint store's total monthly revenue, R, as a function of x_1 and x_2.

Solution 7

The total monthly revenue is the sum of the revenue from each brand:

$$R(x_1, x_2) = x_1 \cdot D_1(x_1, x_2) + x_2 \cdot D_2(x_1, x_2).$$

$$\boxed{R(x_1, x_2) = x_1 D_1(x_1, x_2) + x_2 D_2(x_1, x_2)}$$

This expresses the revenue entirely in terms of x_1, x_2, and the demand functions D_1, D_2.

Problem 8

Compute $\frac{\partial z}{\partial x}$ where $z = \ln xy$.

Solution 8

Note y is treated as a constant.

$$z_x = \frac{y}{xy} = \frac{1}{x}.$$

Problem 9

Compute f_{uv} if $f = uv + e^{u+2v}$

Solution 9

$f_{uv} = (f_u)_v$. Note that

$$f_u = v + 1 \cdot e^{u+2v},$$

implying

$$(f_u)_v = 1 + 2 \cdot e^{u+2v}.$$

Problem 10

Find and classify the critical points of $f(x, y) = (x - 2)^2 + 2y^3 - 6y^2 - 18y + 7$.

Solution 10

We are asked to find and classify the critical points of

$$f(x,y) = (x-2)^2 + 2y^3 - 6y^2 - 18y + 7.$$

Step 1: Compute partial derivatives.

$$f_x = 2(x-2), \qquad f_y = 6y^2 - 12y - 18.$$

Step 2: Solve for critical points.
Set $f_x = 0$:

$$2(x-2) = 0 \quad \Rightarrow \quad x = 2.$$

Set $f_y = 0$:

$$6y^2 - 12y - 18 = 0.$$

Divide by 6:

$$y^2 - 2y - 3 = 0.$$

Factor:

$$(y-3)(y+1) = 0.$$

Thus

$$y = 3, \qquad y = -1.$$

Hence the critical points are:

$$(2,3), \qquad (2,-1).$$

Step 3: Classify using the Hessian.
Second derivatives:

$$f_{xx} = 2, \qquad f_{yy} = 12y - 12, \qquad f_{xy} = 0.$$

The determinant of the Hessian is

$$D = f_{xx}f_{yy} - (f_{xy})^2 = 2(12y - 12).$$

At $(2,3)$:

$$f_{yy}(3) = 12(3) - 12 = 24, \qquad D = 2(24) = 48 > 0.$$

Since $f_{xx} = 2 > 0$, $(2,3)$ is a **local minimum**.
At $(2,-1)$:

$$f_{yy}(-1) = 12(-1) - 12 = -24, \qquad D = 2(-24) = -48 < 0.$$

Thus $(2,-1)$ is a **saddle point**.
Final classification:

$$\boxed{(2,3) \text{ is a local minimum}, \qquad (2,-1) \text{ is a saddle point.}}$$

Problem 11

A manufacturer sells two brands of foot powder, brand A and brand B. When the price of A is x cents per can and the price of B is y cents per can the manufacturer sells $40 - 8x + 5y$ thousand cans of A and $50 + 9x - 7y$ thousand cans of B. The cost to produce A is 10 cents per can and the cost to produce B is 20 cents per can. Determine the selling price of brand A which will maximize the profit.

Solution 11

Let the price of brand A be x cents per can and the price of brand B be y cents per can.

Demand functions:

$$Q_A = 40 - 8x + 5y \quad \text{(in thousands)}, \qquad Q_B = 50 + 9x - 7y \quad \text{(in thousands)}.$$

Profit function.

Revenue:

$$R = xQ_A + yQ_B.$$

Costs:

$$C = 10Q_A + 20Q_B.$$

Thus profit:

$$P(x, y) = R - C = (x - 10)Q_A + (y - 20)Q_B.$$

Substitute Q_A and Q_B:

$$P(x, y) = (x - 10)(40 - 8x + 5y) + (y - 20)(50 + 9x - 7y).$$

Expand each term.

First:

$$(x - 10)(40 - 8x + 5y) = 40x - 8x^2 + 5xy - 400 + 80x - 50y.$$

Second:

$$(y - 20)(50 + 9x - 7y) = 50y + 9xy - 7y^2 - 1000 - 180x + 140y.$$

Add them:

$$P(x, y) = -8x^2 - 7y^2 + (40x + 80x - 180x) + (-50y + 50y + 140y) + (5xy + 9xy) - 400 - 1000.$$

Simplify:

$$P(x, y) = -8x^2 - 7y^2 + (-60x) + 140y + 14xy - 1400.$$

Thus

$$P(x, y) = -8x^2 - 7y^2 + 14xy - 60x + 140y - 1400.$$

Solution 12: Solution Continued...

Step 1: Critical point.
Compute partial derivatives:

$$P_x = -16x + 14y - 60, \qquad P_y = -14y + 14x + 140.$$

Set each to zero:

$$-16x + 14y - 60 = 0,$$
$$-14y + 14x + 140 = 0.$$

Solve the system.
From the second equation:

$$14x - 14y = -140 \quad \Rightarrow \quad x - y = -10 \quad \Rightarrow \quad y = x + 10.$$

Substitute into the first:

$$-16x + 14(x + 10) - 60 = 0,$$
$$-16x + 14x + 140 - 60 = 0,$$
$$-2x + 80 = 0,$$
$$x = 40.$$

Thus

$$y = x + 10 = 50.$$

Step 2: Verify maximum.
Second derivatives:

$$P_{xx} = -16, \qquad P_{yy} = -14, \qquad P_{xy} = 14.$$

Discriminant check:

$$D = P_{xx}P_{yy} - (P_{xy})^2 = (-16)(-14) - 14^2 = 224 - 196 = 28 > 0.$$

Since $P_{xx} = -16 < 0$, the critical point is a **local maximum**.
Conclusion.
The profit is maximized when

$$\boxed{x = 40 \text{ cents per can}}.$$

Problem 12

Skip, not covered in class or final.

Solution 13

Skip.

Problem 13

Skip, not covered in class or final.

Problem 14

Find the maximum value of the function $f(x, y) = 20x^{3/2}y$ subject to the constraint $x + y = 60$. Round your answer to the nearest integer.

Solution 14

We want to maximize

$$f(x,y) = 20x^{3/2}y$$

subject to the constraint

$$x + y = 60.$$

Step 1: Substitute the constraint.
From $x + y = 60$ we have

$$y = 60 - x.$$

Thus

$$F(x) = 20x^{3/2}(60 - x).$$

Step 2: Differentiate.
Let

$$F(x) = 20(60x^{3/2} - x^{5/2}) = 1200x^{3/2} - 20x^{5/2}.$$

Compute derivative:

$$F'(x) = 1200 \cdot \frac{3}{2}x^{1/2} - 20 \cdot \frac{5}{2}x^{3/2} = 1800x^{1/2} - 50x^{3/2}.$$

Factor:

$$F'(x) = 50x^{1/2}(36 - x).$$

Step 3: Critical points.
Set $F'(x) = 0$:

$$50x^{1/2}(36 - x) = 0.$$

Thus

$$x = 0 \quad \text{or} \quad x = 36.$$

Since $x = 0$ gives zero output, the maximum occurs at

$$x = 36.$$

Then

$$y = 60 - 36 = 24.$$

Step 4: Maximum value.
Compute

$$f(36, 24) = 20(36)^{3/2}(24).$$

Note:

$$36^{3/2} = (36^{1/2})^3 = 6^3 = 216.$$

Thus:

$$f(36, 24) = 20 \cdot 216 \cdot 24 = 20 \cdot 5184 = 103680.$$

Rounded to the nearest integer:

$$\boxed{103680}.$$

Problem 15

Evaluate

$$\int_1^2 \int_0^1 (2x + y)\, dydx.$$

Solution 15

We evaluate

$$\int_1^2 \int_0^1 (2x + y)\, dy\, dx.$$

Step 1: Integrate with respect to y.

$$\int_0^1 (2x + y)\, dy = \left[2xy + \frac{y^2}{2}\right]_0^1 = 2x(1) + \frac{1}{2} = 2x + \frac{1}{2}.$$

Step 2: Integrate with respect to x.

$$\int_1^2 \left(2x + \frac{1}{2}\right) dx = \left[x^2 + \frac{x}{2}\right]_1^2.$$

Evaluate:

$$(4 + 1) - \left(1 + \frac{1}{2}\right) = 5 - \frac{3}{2} = \frac{7}{2}.$$

$$\boxed{\frac{7}{2}}.$$

Problem 16

The general solution of

$$\frac{dy}{dx} = 2y + 1$$

is.

Solution 16

We solve the differential equation

$$\frac{dy}{dx} = 2y + 1.$$

Rewrite:

$$\frac{dy}{2y+1} = dx.$$

Integrate both sides.

$$\int \frac{1}{2y+1}\, dy = \int 1\, dx.$$

Left side:

$$\int \frac{1}{2y+1}\, dy = \frac{1}{2} \ln|2y+1|.$$

Thus:

$$\frac{1}{2} \ln|2y+1| = x + C.$$

Multiply by 2:

$$\ln|2y+1| = 2x + C_1.$$

Exponentiate:

$$2y + 1 = Ce^{2x}.$$

Solve for y:

$$y = \frac{Ce^{2x} - 1}{2}.$$

$$\boxed{2y + 1 = Ce^{2x}}$$

is the general solution.

Problem 17

The value, V, of a certain \$1500 IRA account grows at a rate equal to 13.5% of its value. Its value after t years is

Solution 17

The IRA value $V(t)$ grows at a rate proportional to its value:

$$\frac{dV}{dt} = 0.135\,V.$$

This is exponential growth. The general solution is

$$V(t) = Ce^{0.135t}.$$

Use the initial value $V(0) = 1500$:

$$1500 = Ce^0 = C.$$

Thus the value after t years is

$$\boxed{V(t) = 1500e^{0.135t}}.$$

Problem 18

It is estimated that after t years from now the population of a certain town will be increasing at a rate of $5+3t^{2/3}$ hundred per year. If the population is presently $100,000$, by how many people will the population increase over the next 8 years?

Solution 18

The rate of change of the population is given by

$$\frac{dP}{dt} = 5 + 3t^{2/3}$$

(hundreds of people per year).
We want the total increase in population over the next 8 years:

$$\Delta P = \int_0^8 (5 + 3t^{2/3})\,dt,$$

and then multiply by 100 because the rate is in *hundreds*.
Step 1: Integrate.

$$\int_0^8 5\,dt = 5t\Big|_0^8 = 40,$$

$$\int_0^8 3t^{2/3}\,dt = 3 \cdot \frac{3}{5}t^{5/3}\Big|_0^8 = \frac{9}{5} \cdot 8^{5/3}.$$

Compute $8^{5/3}$:

$$8^{1/3} = 2, \qquad 8^{5/3} = 2^5 = 32.$$

Thus:

$$\frac{9}{5} \cdot 32 = \frac{288}{5} = 57.6.$$

Total increase in hundreds:

$$40 + 57.6 = 97.6.$$

Convert to actual people:

$$97.6 \times 100 = 9760.$$

$$\boxed{9760 \text{ people}}$$

is the population increase over the next 8 years.

Problem 19

Calculate the improper integral

$$\int_0^\infty xe^{-x^2}\,dx.$$

Solution 19

We evaluate the improper integral

$$\int_0^\infty xe^{-x^2}\,dx.$$

Step 1: Substitute.
Let

$$u = x^2 \quad \Rightarrow \quad du = 2x\,dx \quad \Rightarrow \quad x\,dx = \frac{1}{2}du.$$

The integral becomes

$$\int_0^\infty xe^{-x^2}\,dx = \frac{1}{2}\int_0^\infty e^{-u}\,du.$$

Step 2: Evaluate.

$$\frac{1}{2}\int_0^\infty e^{-u}\,du = \frac{1}{2}\lim_{N\to\infty}\left[-e^{-u}\right]_0^N = \frac{1}{2}(0-(-1)) = \frac{1}{2}.$$

$$\boxed{\frac{1}{2}}.$$

Problem 20

An object moves so that its velocity after t minutes is given by the formula $v = 20e^{-0.01t}$. The distance it travels during the 10th minute is

Solution 20

$$\int_9^{10} 20e^{-0.01t}\,dt$$

.

Problem 21

Find the sum of the series

$$\sum_{n=1}^\infty \left(-\frac{2}{3}\right)^n.$$

Solution 21

We want to compute the geometric series

$$\sum_{n=1}^{\infty} \left(-\frac{2}{3}\right)^n.$$

Step 1: Shift the index.
Let $k = n - 1$. Then when $n = 1$, $k = 0$; and as $n \to \infty$, $k \to \infty$.
Rewrite the sum:

$$\sum_{n=1}^{\infty} \left(-\frac{2}{3}\right)^n = \sum_{k=0}^{\infty} \left(-\frac{2}{3}\right)^{k+1}.$$

Factor out one power:

$$= \left(-\frac{2}{3}\right) \sum_{k=0}^{\infty} \left(-\frac{2}{3}\right)^k.$$

Step 2: Evaluate the geometric series.
For $|r| < 1$,

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}.$$

Here $r = -\frac{2}{3}$. Thus:

$$\sum_{k=0}^{\infty} \left(-\frac{2}{3}\right)^k = \frac{1}{1 - \left(-\frac{2}{3}\right)} = \frac{1}{1 + \frac{2}{3}} = \frac{1}{\frac{5}{3}} = \frac{3}{5}.$$

Step 3: Multiply by the factored term.

$$\left(-\frac{2}{3}\right)\left(\frac{3}{5}\right) = -\frac{2}{5}.$$

$$\boxed{-\frac{2}{5}}.$$

Problem 22

Use a Taylor polynomial of degree 2 to approximate

$$\int_0^{0.1} \frac{100}{x^2+1}\, dx.$$

Round answer to five decimal places.

Solution 22

We approximate

$$\int_0^{0.1} \frac{100}{1+x^2}\,dx$$

using the degree–2 Taylor polynomial of the integrand about $x = 0$.

$$\frac{100}{1+x^2} = 100(1 - x^2 + x^4 - \cdots) \approx 100(1 - x^2).$$

Now integrate:

$$\int_0^{0.1} 100(1 - x^2)\,dx = 100\left[x - \frac{x^3}{3}\right]_0^{0.1}.$$

Evaluate:

$$100\left(0.1 - \frac{0.001}{3}\right) = 100(0.1 - 0.000333333) = 100(0.099666667) = 9.9666667.$$

Rounded to five decimal places:

$$\boxed{9.96667}.$$

Problem 23

Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{n3^n x^n}{5^{n+1}}.$$

Solution 23

Consider the power series

$$\sum_{n=0}^{\infty} \frac{n3^n x^n}{5^{n+1}} = \frac{1}{5}\sum_{n=0}^{\infty} n\left(\frac{3x}{5}\right)^n.$$

This is a series of the form $\sum nr^n$, which converges when $|r| < 1$.
Here,

$$r = \frac{3x}{5}.$$

Convergence condition:

$$\left|\frac{3x}{5}\right| < 1 \quad \Rightarrow \quad |x| < \frac{5}{3}.$$

Radius of convergence:

$$\boxed{R = \frac{5}{3}}.$$

Problem 24

Find the Taylor series of $f(x) = \dfrac{x}{2+x^2}$ at $x = 0$ (Also called Maclaurin series).

Solution 24

We want the Taylor series of

$$f(x) = \frac{x}{2+x^2} \quad \text{at } x = 0.$$

Step 1: Factor to get geometric series form.

$$f(x) = \frac{x}{2+x^2} = \frac{x}{2} \cdot \frac{1}{1+\frac{x^2}{2}} = \frac{x}{2} \cdot \frac{1}{1-\left(-\frac{x^2}{2}\right)}.$$

Step 2: Use geometric series expansion.

$$\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n \quad \text{for } |r| < 1.$$

Here $r = -\frac{x^2}{2}$. Thus

$$\frac{1}{1-(-x^2/2)} = \sum_{n=0}^{\infty} \left(-\frac{x^2}{2}\right)^n = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^n}.$$

Step 3: Multiply by $x/2$.

$$f(x) = \frac{x}{2} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^n} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2^{n+1}}.$$

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{n+1}}}.$$

Problem 25

Write the following infinite series in summation notation.

$$5 - \frac{7}{8} + \frac{9}{27} - \frac{11}{64} + \cdots$$

Solution 25

Observe the pattern of the series:

$$5, -\frac{7}{8}, \frac{9}{27}, -\frac{11}{64}, \dots$$

Step 1: Numerator pattern.
The numerators are $5, 7, 9, 11, \dots$, which is $2n+3$ if we let $n = 1, 2, 3, \dots$.
Step 2: Denominator pattern.
The denominators are $1, 8, 27, 64, \dots = n^3$. Actually, check:

$$1 = 1^3, \quad 8 = 2^3, \quad 27 = 3^3, \quad 64 = 4^3.$$

Thus, the denominators are n^3.
Step 3: Sign pattern.
The signs alternate: $+, -, +, -, \dots$ This is $(-1)^{n+1}$ for $n = 1, 2, 3, \dots$.
Step 4: Combine.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n+3}{n^3}.$$

$$\boxed{\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n+3}{n^3}}$$

Problem 26

Skip this problem, not on final exam.

Solution 26

Skip.

Problem 27

Find the Taylor series about $x = 0$ for the indefinite integral

$$\int x e^{-x^3}\, dx.$$

Solution 27

We want a Taylor series for

$$\int xe^{-x^3}\,dx$$

about $x = 0$.

Step 1: Recall the Maclaurin series for e^u.

$$e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!}.$$

Take $u = -x^3$:

$$e^{-x^3} = \sum_{n=0}^{\infty} \frac{(-x^3)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{n!}.$$

Step 2: Multiply by x.

$$xe^{-x^3} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+1}}{n!}.$$

Step 3: Integrate term by term.

$$\int xe^{-x^3}\,dx = \sum_{n=0}^{\infty} \int \frac{(-1)^n x^{3n+1}}{n!}dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{x^{3n+2}}{3n+2} + C.$$

$$\boxed{\int xe^{-x^3}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+2}}{(3n+2)n!} + C}.$$

Problem 28

A patient is given an injection of 50 milligrams of a drug every 24 hours. After t days, the fraction of the drug remaining in the patient's body is

$$f(t) = 2^{-t/3}.$$

If the treatment is continued indefinitely, approximately how many milligrams of the drug will eventually be in the patient's body just prior to an injection?

Solution 28

The patient receives 50 mg every 24 hours, and the fraction of drug remaining after t days is

$$f(t) = 2^{-t/3}.$$

We want the total amount in the body just prior to a new injection if treatment continues indefinitely. This is an infinite sum of the form:

$$\text{Total} = 50\,[f(0) + f(1) + f(2) + \cdots] \quad \text{with } t \text{ in days between injections.}$$

Here each term corresponds to the fraction remaining from each previous dose just before the next injection:

$$\text{Total} = 50 \sum_{n=0}^{\infty} 2^{-n/3}.$$

Step 1: Recognize geometric series.
$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ for $|r| < 1$. Here $r = 2^{-1/3}$.

$$\sum_{n=0}^{\infty} 2^{-n/3} = \frac{1}{1 - 2^{-1/3}}.$$

Step 2: Multiply by 50 mg.

$$\text{Total} = \frac{50}{1 - 2^{-1/3}}.$$

Step 3: Approximate numerically.

$$2^{1/3} \approx 1.26 \quad \Rightarrow \quad 2^{-1/3} \approx 0.7937,$$
$$1 - 0.7937 \approx 0.2063,$$
$$\frac{50}{0.2063} \approx 242.4.$$

$$\boxed{\text{Approximately } 242.4 \text{ mg}}.$$

Problem 29: (Problem 32 in Study Guide)

Find the volume of the solid generated by revolving the region bounded by:

$$y = 3e^{2x}, \quad y = 0,\ x = 1,\ \text{ and } x = 3.$$

Solution 29

We use the disk method. The volume is

$$V = \pi \int_1^3 \left(3e^{2x}\right)^2 dx = \pi \int_1^3 9e^{4x}\, dx = 9\pi \int_1^3 e^{4x}\, dx.$$

Step 1: Integrate.

$$\int e^{4x}\, dx = \frac{1}{4}e^{4x}.$$

Step 2: Apply limits.

$$9\pi \int_1^3 e^{4x}\, dx = 9\pi \cdot \frac{1}{4}\left[e^{12} - e^{4}\right] = \frac{9\pi}{4}\left(e^{12} - e^{4}\right).$$

$$\boxed{V = \frac{9\pi}{4}\left(e^{12} - e^{4}\right)}.$$

Problem 30: (Problem 37 in Study Guide)

A nature preserve wishes to construct a large compound which will hold both lions and gazelles. They currently have 6 gazelles. They estimate that if they use an area of A square miles and introduce L lions, they will be able to support a population of G gazelles, given by the function

$$G(A, L) = 6 + 40A - A^2 - 18L^2 + 176L - 8AL.$$

What conditions will lead to the largest number of gazelles?

Solution 30

We are asked to maximize

$$G(A, L) = 6 + 40A - A^2 - 18L^2 + 176L - 8AL$$

with respect to A and L.

Step 1: Compute partial derivatives.

$$G_A = \frac{\partial G}{\partial A} = 40 - 2A - 8L$$

$$G_L = \frac{\partial G}{\partial L} = -36L + 176 - 8A$$

Step 2: Set partial derivatives equal to zero.

$$40 - 2A - 8L = 0 \quad \Rightarrow \quad 2A + 8L = 40 \quad \Rightarrow \quad A + 4L = 20$$

$$-36L - 8A + 176 = 0 \quad \Rightarrow \quad 8A + 36L = 176 \quad \Rightarrow \quad 2A + 9L = 44$$

Step 3: Solve the system of equations.
From the first equation: $A = 20 - 4L$.
Substitute into the second:

$$2(20 - 4L) + 9L = 44 \quad \Rightarrow \quad 40 - 8L + 9L = 44 \quad \Rightarrow \quad L = 4$$

Then

$$A = 20 - 4(4) = 4$$

Step 4: Verify maximum using second partials.

$$G_{AA} = -2, \quad G_{LL} = -36, \quad G_{AL} = -8$$

Compute the determinant of the Hessian:

$$D = G_{AA}G_{LL} - (G_{AL})^2 = (-2)(-36) - (-8)^2 = 72 - 64 = 8 > 0$$

Since $G_{AA} < 0$ and $D > 0$, the critical point is a **maximum**.
Step 5: Maximum conditions.
The largest number of gazelles occurs when

$$\boxed{A = 4 \text{ square miles}, \quad L = 4 \text{ lions}}.$$

Problem 31: (Problem 38 in the Study Guide)

Evaluate

$$\int\int (e^{x^2+1})\, dA$$

where R is the region indicated by the boundaries below:

$$0 \le x \le 1, \quad 0 \le y \le x.$$

Solution 31

We are asked to evaluate

$$\iint_R e^{x^2+1}\, dA$$

where

$$R = \{(x, y) \mid 0 \le x \le 1,\ 0 \le y \le x\}.$$

Step 1: Set up the integral.
Since y varies from 0 to x for each x, we integrate with respect to y first:

$$\iint_R e^{x^2+1}\, dA = \int_0^1 \int_0^x e^{x^2+1}\, dy\, dx.$$

Step 2: Integrate with respect to y.

$$\int_0^x e^{x^2+1}\, dy = e^{x^2+1} \cdot y\Big|_0^x = x\, e^{x^2+1}.$$

Step 3: Integrate with respect to x.

$$\int_0^1 x\, e^{x^2+1}\, dx.$$

Let $u = x^2 + 1 \Rightarrow du = 2x\, dx \Rightarrow x\, dx = \frac{du}{2}$.

$$\int_0^1 xe^{x^2+1}\, dx = \int_{u=1}^2 \frac{1}{2} e^u\, du = \frac{1}{2}\int_1^2 e^u\, du.$$

Step 4: Evaluate the integral.

$$\frac{1}{2}\left[e^u\right]_1^2 = \frac{1}{2}(e^2 - e) = \frac{e^2 - e}{2}.$$

$$\boxed{\frac{e^2 - e}{2}}$$

Problem 32: (Problem 40 in the Study Guide)

Find the general solution to the differential equation

$$-x^5 \sin x + xy' = 3y, \quad x > 0.$$

Solution 32

We are asked to solve

$$-x^5 \sin x + xy' = 3y, \quad x > 0.$$

Step 1: Rewrite in standard linear form.
Divide both sides by x (since $x > 0$):

$$y' - \frac{3}{x}y = x^4 \sin x.$$

This is now in standard form:

$$y' + P(x)y = Q(x), \quad \text{with } P(x) = -\frac{3}{x},\ Q(x) = x^4 \sin x.$$

Step 2: Compute the integrating factor.

$$\mu(x) = e^{\int P(x)dx} = e^{\int -\frac{3}{x}dx} = e^{-3\ln x} = x^{-3}.$$

Step 3: Multiply both sides by the integrating factor.

$$x^{-3}y' - \frac{3}{x}x^{-3}y = x^{-3}x^4 \sin x \quad \Rightarrow \quad x^{-3}y' - 3x^{-4}y = x \sin x.$$

Step 4: Left-hand side is the derivative of μy.

$$\frac{d}{dx}\left(x^{-3}y\right) = x \sin x.$$

Step 5: Integrate both sides.

$$x^{-3}y = \int x \sin x \, dx + C.$$

Step 6: Evaluate $\int x \sin x \, dx$ using integration by parts.
Let $u = x \Rightarrow du = dx$, $dv = \sin x dx \Rightarrow v = -\cos x$:

$$\int x \sin x \, dx = -x \cos x + \int \cos x \, dx = -x \cos x + \sin x.$$

Step 7: Solve for y.

$$x^{-3}y = -x \cos x + \sin x + C \quad \Rightarrow \quad y = x^3(-x \cos x + \sin x + C).$$

Final Answer:

$$\boxed{y = x^3(\sin x - x \cos x + C)}$$