HWK 4 – Problem 9

Problem: Consider the ellipse given by the equation:

$$\frac{x^2}{36} + \frac{y^2}{81} = 1$$

We are asked to find the volume of the solid generated when this ellipse is rotated:

(a) about the **x-axis**

(b) about the **y-axis**

These solids are called ellipsoids; one is vaguely rugby-ball shaped, one is sort of flying-saucer shaped, or perhaps squished-beach-ball-shaped.

Solution

The given equation represents an ellipse centered at the origin, with:

$$a^2 = 36 \Rightarrow a = 6, \quad b^2 = 81 \Rightarrow b = 9$$

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(a) Rotation About the x-axis

We solve for y in terms of x:

$$\frac{x^2}{36} + \frac{y^2}{81} = 1 \quad \Rightarrow \quad y^2 = 81\left(1 - \frac{x^2}{36}\right)$$

We apply the disk method:

$$V(x) = \pi \int_{-6}^{6} y^2 \, dx = \pi \int_{-6}^{6} 81\left(1 - \frac{x^2}{36}\right) dx$$

Factor out constants:

$$V(x) = 81\pi \int_{-6}^{6} \left(1 - \frac{x^2}{36}\right) dx$$

Since the integrand is even, we can simplify:

$$V(x) = 2 \cdot 81\pi \int_0^6 \left(1 - \frac{x^2}{36}\right) dx$$

Compute the integral:

$$\int_0^6 \left(1 - \frac{x^2}{36}\right) dx = \left[x - \frac{x^3}{108}\right]_0^6 = 6 - \frac{216}{108} = 6 - 2 = 4$$

Thus,

$$V(x) = 2 \cdot 81\pi \cdot 4 = 648\pi$$

Decimal approximation:

$$V(x) \approx \boxed{2035.752}$$

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(b) Rotation About the y-axis

We solve for x in terms of y:

$$\frac{x^2}{36} + \frac{y^2}{81} = 1 \quad \Rightarrow \quad x^2 = 36\left(1 - \frac{y^2}{81}\right)$$

Apply the disk method:

$$V_y = \pi \int_{-9}^{9} x^2 \, dy = \pi \int_{-9}^{9} 36 \left(1 - \frac{y^2}{81}\right) dy$$

Factor out constants:

$$V_y = 36\pi \int_{-9}^{9} \left(1 - \frac{y^2}{81}\right) dy = 2 \cdot 36\pi \int_0^9 \left(1 - \frac{y^2}{81}\right) dy$$

Compute the integral:

$$\int_0^9 \left(1 - \frac{y^2}{81}\right) dy = \left[y - \frac{y^3}{243}\right]_0^9 = 9 - \frac{729}{243} = 9 - 3 = 6$$

Thus,

$$V_y = 2 \cdot 36\pi \cdot 6 = 432\pi$$

Decimal approximation:

$$V_y \approx 432 \times 3.141593 \approx \boxed{1357.960}$$

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Final Answers

Volumes of Solids of Revolution

$$\text{Volume about x-axis:} \quad V_x = 648\pi \approx \boxed{2035.752}$$

$$\text{Volume about y-axis:} \quad V_y = 432\pi \approx \boxed{1357.960}$$