MA 16020 – Applied Calculus II: Lecture 19
Separable Differential Equations

What is a Differential Equation?

Definition

A **differential equation** (DE) is an equation that involves an **unknown function** and one or more of its **derivatives**.

Differential equations are used to model how quantities change with respect to one another.

- Example: how a population grows over time,
- how an object moves under a force,
- how temperature changes in a cooling process.

Example 1: Recognizing Differential Equations

Each of the following is a differential equation:

$$\frac{dy}{dt} = 8y, \quad y' = t\cos y, \quad y' = x^3y + xy^2, \quad \frac{dy}{dt} = (\cos t)y + t^2 + \frac{1}{3}y$$

- The variable on the right-hand side can depend on x, t, or y.
- The goal in solving a DE is to find a function $y(x)$ or $y(t)$ that satisfies the equation.
- Differential equations are often classified by *order* (highest derivative) and *type* (linear, separable, etc.).

Example 2: Newton's Second Law of Motion

Newton's Second Law states that

$$F = ma = m\frac{d^2x}{dt^2}.$$

If the force depends on position x, we obtain a second-order DE:

$$m\frac{d^2x}{dt^2} = F(x).$$

Example: A mass attached to a spring with stiffness k obeys

$$m\frac{d^2x}{dt^2} = -kx,$$

which models simple harmonic motion (oscillations).

Example 3: Population Growth in Biology

A common biological model is the **logistic growth equation**:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right),$$

where

- $P(t)$ is the population at time t,
- k is the growth rate,
- M is the carrying capacity (maximum sustainable population).

This is a **first-order separable** differential equation that models population growth limited by available resources.

Order of a Differential Equation

Definition (Intuitive)

The **order** of a differential equation is the highest derivative of the unknown function that appears in it.

Why it matters:

- The order tells us how many **initial conditions** are needed to find a specific solution.
- It also tells us how "dynamic" the system is — for example, position and acceleration lead to a second-order equation.

Examples:

$y' = 3x^2$	First-order
$y'' + 4y' + 4y = 0$	Second-order
$\dfrac{d^3y}{dx^3} = \sin x$	Third-order

Separable Differential Equations

Definition

A **separable differential equation** is one that can be written in the form

$$\frac{dy}{dx} = g(x)\,h(y),$$

where the variables x and y can be separated onto opposite sides of the equation.

Intuitively, this means that the rate of change of y depends on a product of a function of x and a function of y — so we can "separate" their effects.

Example 4: Separable DEs

Examples:

(1) $\dfrac{dy}{dx} = 2x(1+y^2)$ $\Rightarrow$separable: $\dfrac{dy}{1+y^2} = 2x\,dx$

(2) $\dfrac{dy}{dt} = te^y$ $\Rightarrow$separable: $e^{-y}\,dy = t\,dt$

(3) $\dfrac{dy}{dx} = y + x$ $\Rightarrow$not separable (sum, not product).

Recognizing a Separable Differential Equation

We want to write a differential equation in the form

$$\frac{dy}{dx} = (\text{function of } x) \times (\text{function of } y)$$

so that the variables x and y can be separated onto opposite sides of the equation.

Example 5. Determine whether the following equation is separable:

$$\frac{dy}{dx} = x^2 e^{3y-2x^2}.$$

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Example 5. Determine whether the following equation is separable:

$$\frac{dy}{dx} = x^2 e^{3y - 2x^2}.$$

Solution (conceptual only):

$$e^{3y-2x^2} = e^{3y}\, e^{-2x^2},$$

so

$$\frac{dy}{dx} = \left(x^2 e^{-2x^2}\right)\left(e^{3y}\right).$$

This is of the form $g(x)\, h(y)$, so the equation is **separable.**

Checking a Solution to a Differential Equation

Before learning how to *solve* separable differential equations, let's see how to *check* whether a given function satisfies one.

Example 6. Verify that $y = \sqrt{x^2 - 17}$ is a solution to

$$\frac{dy}{dx} = \frac{x}{y}.$$

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Solution:

$$\frac{dy}{dx} = \frac{1}{2}(x^2 - 17)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 - 17}}.$$

Since $y = \sqrt{x^2 - 17}$, we have

$$\frac{dy}{dx} = \frac{x}{y}.$$

Therefore, $y = \sqrt{x^2 - 17}$ **satisfies** the differential equation.

Always check by differentiating the proposed solution and substituting it back into the equation.

Idea: Separation of Variables

The goal in solving a **separable differential equation** is to **separate the variables** so that all y-terms are on one side and all x-terms on the other.

General form:

$$\frac{dy}{dx} = g(x)\,h(y)$$

If $h(y) \neq 0$, we can rearrange terms:

$$\frac{1}{h(y)}\,dy = g(x)\,dx$$

Now each side depends on only one variable, so we can integrate both sides:

$$\int \frac{1}{h(y)}\,dy = \int g(x)\,dx$$

Particular Solutions of Separable Differential Equations

When we solve a differential equation, we usually obtain a **family of solutions** containing an arbitrary constant C:

$$y = f(x, C)$$

A **particular solution** is obtained by using an **initial condition** to find the specific value of C.

Example 7:

1. Find the particular solution to

$$\frac{dy}{dx} = \frac{x^2}{y^2}, \quad \text{if } y = 1 \text{ when } x = 0.$$

2. Find the particular solution to

$$y' = ky, \quad \text{given } y(0) = 12,\ y'(0) = 24.$$

Example 8: General Solution

Example 8. Find the **general solution** to the differential equation

$$\frac{dy}{dt} = k(50 - y),$$

assuming $50 - y > 0$.

Example 9: Radioactive Decay and Half-Life

Suppose $P(t)$ is the mass of a radioactive substance at time t. The differential equation is

$$P'(t) = -\frac{3}{2}P(t).$$

Instructions:

- Find the **half-life** t such that $P(t) = \frac{A}{2}$ where A is the amount at time 0.

Half-Life Definition

The **half-life** of a substance is the amount of time it takes for half the substance to disappear.