# MA 16020 – Applied Calculus II: Lecture 4: Integration by Substitution II

# Integration by Substitution

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The **Chain Rule**:

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$$

Reversing the chain rule gives a method for integrals:

$$\int f'(g(x)) \cdot g'(x)\, dx = f(g(x)) + C$$

**Idea:** Let $u = g(x)$, so that $du = g'(x)\, dx$. Then:

$$\int f'(g(x)) \cdot g'(x)\, dx = \int f'(u)\, du$$

$$= f(u) + C = f(g(x)) + C$$

# Lecture 4: Example 1 – Practice Integrals

We now practice more challenging u-sub problems. Compute the following antiderivatives using $u$-substitution:

(a) $\displaystyle\int \frac{5x}{\sqrt{x+6}}\,dx$

(b) $\displaystyle\int x^2\sqrt{2x-5}\,dx$

# Example 1(a) – Step 1: Rewrite and Substitution

Compute $\displaystyle\int \frac{5x}{\sqrt{x+6}}\,dx$.

- Pick substitution: $u = x + 6 \quad \Rightarrow \quad du = dx$
- Rewrite $x$ in terms of $u$: $x = u - 6$
- Substitute into integral:

$$\int \frac{5x}{\sqrt{x+6}}\,dx = \int \frac{5(u-6)}{\sqrt{u}}\,du$$

- Simplify the integrand using exponents:

$$\int \frac{5(u-6)}{u^{1/2}}\,du = \int 5(u^{1/2} - 6u^{-1/2})\,du$$

# Example 1(a) – Step 2: Integrate and Final Answer

- From previous slide we had

$$\int \frac{5x}{\sqrt{x+6}}\,dx = \int 5\left(u^{1/2} - 6u^{-1/2}\right)du, \quad u = x+6.$$

- Integrate term-by-term:

$$\int 5u^{1/2}\,du = \frac{10}{3}u^{3/2}, \qquad \int -30u^{-1/2}\,du = -60u^{1/2}.$$

- Combine and back-substitute $u = x+6$:

$$F(x) = \frac{10}{3}(x+6)^{3/2} - 60(x+6)^{1/2} + C.$$

- (Optional tidy form)

$$F(x) = \frac{10}{3}(x+6)^{1/2}\left(x-12\right) + C.$$

# Example 1(b) – Setup

- Compute
$$\int x^2\sqrt{2x-5}\,dx.$$
- Let
$$u = 2x-5 \quad \Rightarrow \quad du = 2\,dx, \quad dx = \tfrac{1}{2}du.$$
- Express $x$ in terms of $u$:
$$x = \tfrac{u+5}{2}, \quad x^2 = \left(\tfrac{u+5}{2}\right)^2.$$
- Substitute:
$$\int x^2\sqrt{2x-5}\,dx = \tfrac{1}{2}\int \left(\tfrac{u+5}{2}\right)^2 u^{1/2}\,du.$$
- (Stop here – solve on board.)

# Example 2 – FTC with $u$-Substitution

Compute the following definite integrals by $u$-substitution and carefully change the bounds.

(a) $\displaystyle\int_0^1 6x\,(x^2-1)^2\,dx.$

- Suggestion: let $u = x^2 - 1 \Rightarrow du = 2x\,dx$.
- Then $6x\,dx = 3\,du$. Bounds: $x = 0 \Rightarrow u = -1,\ x = 1 \Rightarrow u = 0$.

(b) $\displaystyle\int_{-1}^2 24(x^2-2)\,(x^3-6x)^4\,dx.$

- Suggestion: let $u = x^3 - 6x \Rightarrow du = (3x^2 - 6)\,dx = 3(x^2-2)\,dx$.

**Key idea:** Always transform the bounds to the $u$-variable (don't revert to $x$), or record both forms and evaluate directly.

# Average Value of a Function

Average value of a function over an interval

If $f(x)$ is defined on an interval $[a, b]$, then the average value of $f(x)$ over $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x)\, dx$$

# Example: Average Value

Find the average value of the function

$$f(x) = 4.4xe^{x^2}$$

over the interval $0 < x < 1.8$. Round your answer to the nearest hundredth.

# Example: Step 1 – $u$-Substitution

Compute

$$\int_0^{1.8} 4.4xe^{x^2}\,dx.$$

- Let $u = x^2 \quad \Rightarrow \quad du = 2x\,dx$
- Then $4.4x\,dx = 2.2\,du$
- Rewrite the integral in terms of $u$:

$$\int_0^{1.8} 4.4xe^{x^2}\,dx = \int_0^{(1.8)^2} 2.2e^u\,du = 2.2\int_0^{3.24} e^u\,du$$

# Example: Step 2 – Evaluate and Find Average

- Evaluate the integral:

$$2.2 \int_0^{3.24} e^u \, du = 2.2 \left[e^u\right]_0^{3.24} = 2.2(e^{3.24} - 1)$$

- Divide by the interval length to find average value:

$$f_{\text{avg}} = \frac{1}{1.8 - 0} \cdot 2.2(e^{3.24} - 1) = \frac{2.2}{1.8}(e^{3.24} - 1)$$

- Approximate to nearest hundredth:

$$f_{\text{avg}} \approx 28.38$$