MA 16020 – Applied Calculus II: Lecture 3: Integration by Substitution I

Warm-Up: Identifying Inner and Outer Functions

Determine the inner and outer functions for the following. List the outer function as $f(x)$ and the inner function as $g(x)$, then check $f(g(x))$.

a. $h(x) = \tan(3x^2)$
b. $h(x) = 4e^{2x}$
c. $h(x) = \sqrt[3]{3x^2 + 2}$

Warm-Up Solutions

Identify $f(x)$ (outer) and $g(x)$ (inner). Check $f(g(x)) = h(x)$.

- (a) $h(x) = \tan(3x^2)$ $\quad f(x) = \tan(x)$, $\quad g(x) = 3x^2$
  $f(g(x)) = \tan(3x^2)$
- (b) $h(x) = 4e^{2x}$ $\quad f(x) = 4e^x$, $\quad g(x) = 2x$ $\quad f(g(x)) = 4e^{2x}$
- (c) $h(x) = \sqrt[3]{3x^2 + 2}$ $\quad f(x) = \sqrt[3]{x}$, $\quad g(x) = 3x^2 + 2$
  $f(g(x)) = \sqrt[3]{3x^2 + 2}$

Chain Rule

- Suppose $y = f(g(x))$, i.e. y is a composition of two functions.
- **Newton notation:**

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

- **Leibniz notation:**

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

  where $u = g(x)$ and $y = f(u)$.
- Interpretation: Differentiate the *outer function* (leaving the inside intact), then multiply by the derivative of the *inner function*.

Example: Chain Rule in Action

- Let $y = \sin(x^2)$.
- **Newton notation:**

$$\frac{d}{dx}\left(\sin(x^2)\right) = \cos(x^2)\cdot(2x) = 2x\cos(x^2).$$

- **Leibniz notation:**

$$y = \sin(u), \quad u = x^2,$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \cos(u)\cdot(2x) = 2x\cos(x^2).$$

Integration by Substitution

The **Chain Rule**:

$$\frac{d}{dx}\big(f(g(x))\big) = f'(g(x)) \cdot g'(x)$$

Integration by Substitution

The **Chain Rule**:

$$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x)$$

Reversing the chain rule gives a method for integrals:

$$\int f'(g(x)) \cdot g'(x)\, dx = f(g(x)) + C$$

Integration by Substitution

The **Chain Rule**:

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$$

Reversing the chain rule gives a method for integrals:

$$\int f'(g(x)) \cdot g'(x)\, dx = f(g(x)) + C$$

Idea: Let $u = g(x)$, so that $du = g'(x)\, dx$. Then:

$$\int f'(g(x)) \cdot g'(x)\, dx = \int f'(u)\, du$$

$$= f(u) + C = f(g(x)) + C$$

Example 1: $\int 2x\cos(x^2)\,dx$

1. **Choose substitution:** $u = x^2$.
2. **Differentiate:** $du = 2x\,dx \Rightarrow 2x\,dx = du$.
3. **Substitute:**

$$\int 2x\cos(x^2)\,dx = \int \cos(u)\,du.$$

4. **Integrate:**

$$\int \cos(u)\,du = \sin(u) + C.$$

5. **Back-substitute:**

$$\sin(u) + C = \sin(x^2) + C.$$

Remark: this is the chain rule in reverse since $\frac{d}{dx}\left(\sin(x^2)\right) = 2x\cos(x^2)$.

Example 2: Practice Problems

Directions: Find the antiderivatives of the following integrals. Use *u*-substitution.

(a) $\displaystyle\int 4e^{2x}\,dx$

(b) $\displaystyle\int \sqrt{4x+1}\,dx$

(c) $\displaystyle\int \frac{3\sin x}{\cos^8 x}\,dx$

(d) $\displaystyle\int x\sqrt{10-2x^2}\,dx$

Example 2(a)

Compute $\int 4e^{2x}\,dx$.

- Let $u = 2x \quad \Rightarrow \quad du = 2\,dx$.
- Then $dx = \frac{1}{2}\,du$.
- Substitute:

$$\int 4e^{2x}\,dx = \int 4e^{u} \cdot \tfrac{1}{2}\,du = 2\int e^{u}\,du$$

- Antiderivative:

$$2e^{u} + C = 2e^{2x} + C$$

Example 2(b)

Compute $\int \sqrt{4x+1}\,dx$.

- Rewrite: $\sqrt{4x+1} = (4x+1)^{1/2}$.
- Let $u = 4x+1 \quad \Rightarrow \quad du = 4\,dx$.
- Then $dx = \frac{1}{4}\,du$.
- Substitute:

$$\int (4x+1)^{1/2}\,dx = \frac{1}{4}\int u^{1/2}\,du$$

- Antiderivative:

$$\frac{1}{4}\cdot\frac{2}{3}u^{3/2} + C = \frac{1}{6}(4x+1)^{3/2} + C$$

Example 2(c)

Compute $\int \frac{3\sin x}{\cos^8 x}\,dx$.

- Rewrite:
$$\frac{3\sin x}{\cos^8 x} = 3\sin x \cdot \cos^{-8} x.$$
- Let $u = \cos x \quad \Rightarrow \quad du = -\sin x\,dx$.
- Then $\sin x\,dx = -du$.
- Substitute:
$$\int 3\sin x \cdot \cos^{-8} x\,dx = -3\int u^{-8}\,du$$
- Antiderivative:
$$-3 \cdot \frac{u^{-7}}{-7} + C = \frac{3}{7}u^{-7} + C = \frac{3}{7}\cos^{-7} x + C$$

Example 2(d)

Compute $\int x\sqrt{10-2x^2}\,dx$.

- Inside the square root: $10-2x^2$.
- Let $u = 10-2x^2 \quad \Rightarrow \quad du = -4x\,dx$.
- Then $x\,dx = -\frac{1}{4}du$.
- Substitution:

$$\int x\sqrt{10-2x^2}\,dx = \int \sqrt{u}\left(-\tfrac{1}{4}du\right) = -\tfrac{1}{4}\int u^{1/2}\,du$$

- Antiderivative:

$$-\tfrac{1}{4}\cdot\frac{2}{3}u^{3/2}+C = -\tfrac{1}{6}(10-2x^2)^{3/2}+C$$

Example 3

Compute the antiderivatives:

(a) $\int 5x^2 e^{x^3}\, dx$

(b) $\int 6e^{\tan(13x)} \sec^2(13x)\, dx$

Example 3(a)

$$\int 5x^2 e^{x^3}\, dx$$

- Inside the exponential: x^3.
- Let $u = x^3 \quad \Rightarrow \quad du = 3x^2\, dx$.
- Then $5x^2 dx = \frac{5}{3} du$.
- Substitution:

$$\int 5x^2 e^{x^3}\, dx = \frac{5}{3} \int e^u\, du = \frac{5}{3} e^u + C$$

- Back-substitute: $\frac{5}{3} e^{x^3} + C$.

Example 3(b)

$$\int 6e^{\tan(13x)}\sec^2(13x)\,dx$$

- Inside the exponential: $\tan(13x)$.
- Let $u = \tan(13x) \quad \Rightarrow \quad du = 13\sec^2(13x)\,dx$.
- Then $\sec^2(13x)\,dx = \frac{1}{13}du$.
- Substitution:

$$\int 6e^{\tan(13x)}\sec^2(13x)\,dx = \frac{6}{13}\int e^u\,du = \frac{6}{13}e^u + C$$

- Back-substitute: $\frac{6}{13}e^{\tan(13x)} + C$.

Example 4

Find a function $f(x)$ such that:

- The slope of the tangent line is

$$f'(x) = \frac{2 + \sqrt[3]{x}}{3 \cdot (\sqrt[3]{x})^2}, \quad x \neq 0$$

- The graph passes through the point $(1, 5/2)$.

Goal: Determine $f(x)$ by integrating $f'(x)$ and using the given point to find the constant of integration.

Example 4: Step 1 – Set Up the Substitution

Given

$$f'(x) = \frac{2 + \sqrt[3]{x}}{3(\sqrt[3]{x})^2}, \quad x \neq 0$$

- Rewrite as a single line using negative exponents:

$$f'(x)\,dx = \left(2 + x^{1/3}\right) \cdot \frac{1}{3} x^{-2/3}\,dx$$

- Pick substitution: $u = 2 + x^{1/3} \quad \Rightarrow \quad du = \frac{1}{3} x^{-2/3}\,dx$
- Rewrite the integral in terms of u:

$$\int f'(x)\,dx = \int u\,du$$

Example 4: Step 2 – Integrate and Solve for C

- Integrate in terms of u:

$$\int u\,du = \frac{1}{2}u^2 + C$$

- Substitute back $u = 2 + x^{1/3}$:

$$f(x) = \frac{1}{2}\left(2 + x^{1/3}\right)^2 + C$$

- Use the point $(1, 5/2)$ to solve for C:

$$f(1) = \frac{1}{2}(2+1)^2 + C = \frac{1}{2}\cdot 9 + C = \frac{9}{2} + C$$

$$\frac{9}{2} + C = \frac{5}{2} \quad \Rightarrow \quad C = -2$$

- Final solution:

$$\boxed{f(x) = \frac{1}{2}(2 + x^{1/3})^2 - 2}$$