Lecture 2. Review of Basic Integration

Differentiation		Integration
$\frac{d}{dx}$, f'	$\xleftrightarrow{FTC}$	$\int f(x)\,dx$, $\int_a^b f(x)\,dx$
tangent lines		area under a curve
how a function changes		how a function accumulates

Theorem 1 Fundamental Theorem of Calculus (FTC)

(a) $\int_a^b f'(x)\,dx = f(b) - f(a)$

(b) $\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$

FTC tells us derivatives and integrals are "opposites"

Integrals come in two flavors: indefinite and definite

Indefinite integrals

denoted $\int f(x)\,dx$ is list of all functions

$F(x)$ with $\boxed{F'(x) = f(x)}$

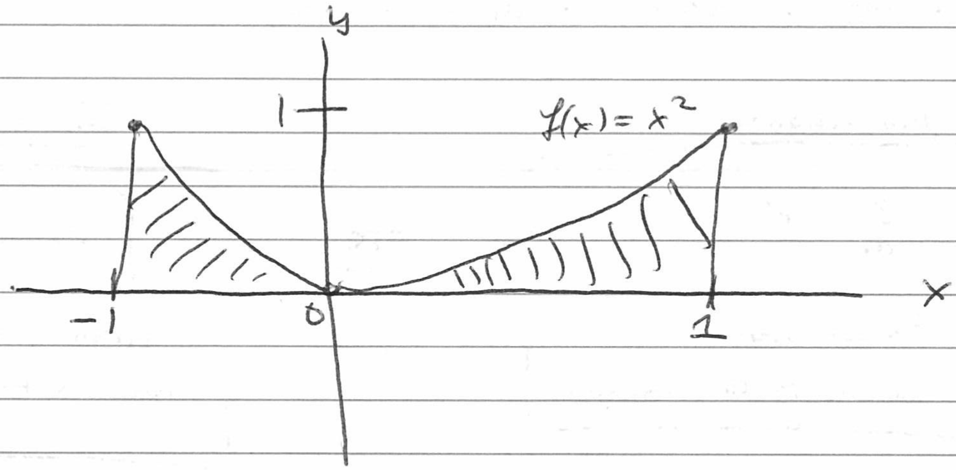
All such $F(x)$ differ by a constant, $+C$.

$\quad\;\, \|$

antiderivative

Definite integral is a number that describes area under a curve under a specified interval.

Ex. 1



The definite integral $\int_{-1}^{1} x^2 \, dx$

For definite integral, you need bounds. Bounds are $x = -1$ and $x = 1$: $[-1, 1]$.

Basic Integration

Examples 1

(1) Evaluate $\int 7x^3 \, dx$ and $\int_1^2 7x^3 \, dx$.

Solution

$$\int 7x^3 \, dx = 7 \int x^3 \, dx = 7 \left(\frac{1}{3+1} x^{3+1} \right) + C$$

★ For indefinite integrals, always add a $+C$

$$= 7 \left(\frac{1}{4} \right) x^4 + C$$

$$= \boxed{\frac{7}{4} x^4 + C}$$

$$\int_1^3 7x^3 \, dx = 7 \int_1^2 x^3 \, dx$$

$$= 7 \left(\frac{1}{3+1} \right) x^{3+1} \Big|_1^2 = 7 \left(\frac{1}{4} \right) x^4 \Big|_1^2$$

$$= \frac{7}{4} x^4 \Big|_1^2$$

$$= \frac{7}{4}\left[(2)^4 - (1)^4\right] = \frac{7}{4}[16-1]$$

$$= \frac{7}{4}(15) = \boxed{\frac{105}{4}}$$

② Evaluate $\int \frac{1}{2}\sec^2 x \, dx$ and $\int_0^{\pi/3} \frac{1}{2}\sec^2 x \, dx$.

Solution. Trig functions will show up frequently in this class.

Remember: $\frac{d}{dx}[\tan x] = \sec^2 x$.

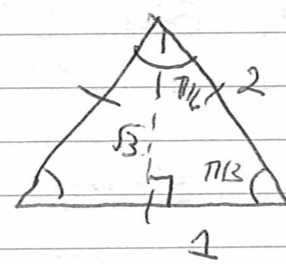
So

$$\int \frac{1}{2}\sec^2 x \, dx = \frac{1}{2}\int \sec^2 x \, dx = \boxed{\frac{1}{2}\tan x + C}$$

$$\int_0^{\pi/3} \frac{1}{2}\sec^2 x \, dx = \frac{1}{2}\int_0^{\pi/3} \sec^2 x \, dx = \frac{1}{2}\tan x \Big|_0^{\pi/3}$$

$$= \frac{1}{2}\tan(\pi/3) - \frac{1}{2}\tan(0)$$

$$= \frac{1}{2}\sqrt{3} - \frac{1}{2}(0) = \boxed{\frac{\sqrt{3}}{2}}$$



$1^2 + ?^2 = 2^2$

$?^2 = 3 \Rightarrow \sqrt{3} = ?$

$\tan\frac{\pi}{3} = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{3}}{1} = \sqrt{3}$

(3) Find $\int (x-1)^2 \, dx$.

Solution. We need to FOIL the function.

Write $(x-1)^2 = (x-1)(x-1) = x^2 + (-1)(x) + (-1)(x) + (-1)(-1)$
$= x^2 - x - x + 1$
$= x^2 - 2x + 1$

Observe. $(x-1)^2 \neq x^2 + 1$

Write $\int (x-1)^2 \, dx = \int x^2 - 2x + 1 \, dx$

$= \int x^2 \, dx + \int (-2x) \, dx + \int 1 \, dx$

$= \frac{1}{2+1} x^{2+1} - \frac{2}{1+1} x^{1+1} + x + C$

$= \frac{1}{3} x^3 - \frac{2}{2} x^2 + x + C = \boxed{\frac{1}{3} x^3 - x^2 + x + C}$

(4) Find $\int_1^4 \frac{x^4 + \sqrt{x^3}}{\sqrt{x}} \, dx$

Solution ☆ Turn roots into fractional exponents

$\sqrt{x^3} = (x^3)^{1/2} = x^{3/2}$

$\therefore$

$\frac{x^4 + \sqrt{x^3}}{\sqrt{x}} = \frac{x^4 + x^{3/2}}{x^{1/2}} = \frac{x^4}{x^{1/2}} + \frac{x^{3/2}}{x^{1/2}}$

$= x^{4-\frac{1}{2}} + x^{3/2 - 1/2} = x^{7/2} + x^{2/2}$

$= x^{7/2} + x^1$

Volume of water $= \int_0^{time} 7\sqrt{x}\,dx$

$||$

118

(2) How many hours after 9:00 am will there be 118 cubic feet of water in the tank?

Solution: Inflow rate is $r(t) = 7\sqrt{t}$ ft^3/hr

$$V(t) = \cancel{\int_0^t 7\sqrt{x}\,dx} \quad \int_0^t 7\sqrt{x}\,dx = 7\int_0^t \sqrt{x}\,dx$$

$$= 7\int_0^t x^{1/2}\,dx = 7\left[\frac{2}{3}x^{3/2}\right]_0^t$$

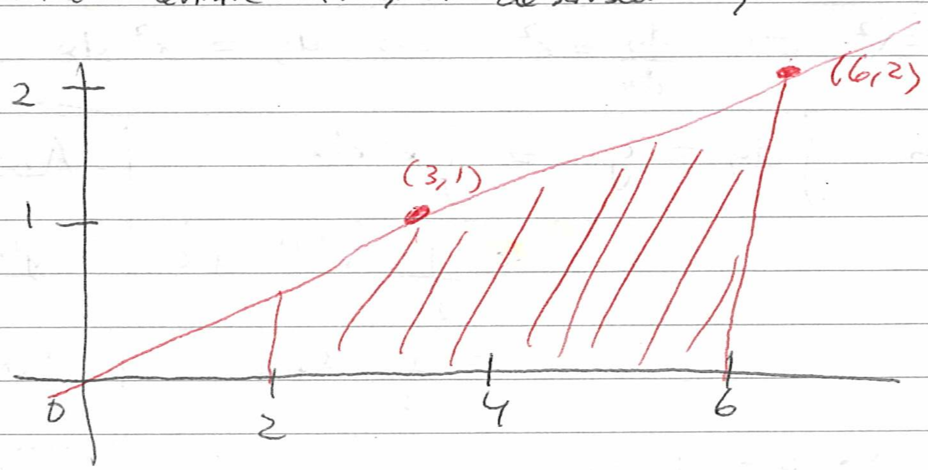
$$= \frac{14}{3}t^{3/2} = 118$$

So $\frac{14}{3}t^{3/2} = 118$

$$\Rightarrow t^{3/2} = \frac{118\cdot 3}{14} = \frac{354}{14} = \frac{177}{7} \approx 25.28$$

$$\Rightarrow (t^{3/2})^{2/3} = \left(\frac{177}{7}\right)^{2/3} \approx \boxed{8.6 \text{ hours}}$$

[3] Find the definite integral described by the following.



Solution

We need two things: (1) the function and (2) the bounds.

(1) Line passes through (3,1) and (6,2).

Slope $m = \frac{2-1}{6-3} = \frac{1}{3}$. line passes through point (0,0).

y-int: $0 = b$

$$y = mx + b = \frac{1}{3}x + 0 = \frac{1}{3}x.$$

(2) Bounds are $x = 2$ and $x = 6$. So definite integral is

$$\boxed{\int_2^6 \frac{1}{3}x\,dx}$$

[4] Find $y(2)$ if $y' = x^2$ and $y(1) = -1$.

Solution $\begin{cases} y' = x^2 \\ y(1) = -1 \end{cases}$ is called a <u>differential equation</u>

$y(1) = -1$ is called an <u>initial condition</u>

This is an integration problem in disguise!

$$y' = x^2 \iff \frac{dy}{dx} = x^2 \iff dy = x^2\,dx$$

then $\int dy = y = \int x^2\,dx$

$$= \frac{1}{2+1}x^{2+1} + C$$

$$= \frac{1}{3}x^3 + C$$

So $y = \frac{1}{3}x^3 + C$.

Find constant C. $y(1) = -1 = \frac{1}{3}(1)^3 + C$

$\Rightarrow -1 = \frac{1}{3} + C \Rightarrow \boxed{C = -1 - \frac{1}{3} = -\frac{4}{3}}$

Answer $\boxed{y = \frac{1}{3}x^3 - \frac{4}{3}}$

$y(2) = \frac{1}{3}(2)^3 - \frac{4}{3}$

$= \frac{1}{3}(8) - \frac{4}{3} = \boxed{\frac{4}{3}}$

So

$$\int_1^4 \frac{x^4+\sqrt{x^3}}{\sqrt{x}} = \int_1^4 \left(x^{7/2}+x\right) dx$$

$$= \frac{1}{7/2+1} x^{7/2+1} + \frac{1}{1+1} x^{1+1} \Big|_1^4$$

$$= \frac{1}{9/2} x^{9/2} + \frac{1}{2} x^2 \Big|_1^4$$

$$= \left(\frac{2}{9}(4)^{9/2} + \frac{1}{2}(4)^2\right) (-) \left(\frac{1}{9/2} x + \frac{1}{2}\right)$$

$$= \left(\frac{2}{9}(2)^{2\cdot\frac{9}{2}} + \frac{1}{2}(16)\right) (-) \left(\frac{2}{9}+\frac{1}{2}\right)$$

$$= \frac{2}{9}(2)^9 + 8 - \frac{2}{9} - \frac{1}{2}$$

$$= \boxed{\frac{2179}{18}}$$

Additional Examples 2. [Applied Calculus means word problems!)

(1) A faucet is turned on at 9:00 am and water starts to flow into a tank at the rate of

$$\boxed{r(t) = 7\sqrt{t}}$$

where t is time in hours after 9:00 am and rate $r(t)$ is in cubic feet per hour.

How much water, in cubic feet, flows into the tank from 10:00 am to 1:00 pm? Give an exact answer.

Solution

We start from 9:00 am so want

$$\text{Volume} = \int_{t_1}^{t_2} r(t)\, dt$$

9:00 am - 10:00 am is 1 hour so $t_1 = 1$
9:00 am - 1:00 pm is 4 hours so $t_2 = 4$

$\therefore$

$$\text{Volume} = \int_1^4 r(t)\, dt = \int_1^4 7\sqrt{t}\, dt = 7 \cdot \int_1^4 \sqrt{t}\, dt$$

$$= 7 \int_1^4 t^{1/2}\, dt$$

$$= 7 \left[\frac{1}{1+\frac{1}{2}} t^{1/2+1} \right]_1^4$$

$$= 7 \left[\frac{2}{3} t^{3/2} \right]_1^4$$

$$= 7 \left[\frac{2}{3} (4)^{3/2} - \frac{2}{3} \right]$$

$$= 7 \left[\frac{2}{3} (2^2)^{3/2} - \frac{2}{3} \right]$$

$$= 7 \left[\frac{2}{3} (2)^3 - \frac{2}{3} \right]$$

$$= 7 \left[\frac{16}{3} - \frac{2}{3} \right] = \frac{7}{3} [16-2] = \frac{7}{3}(14) = \boxed{\frac{98}{3}\ ft^3}$$

Answer: $\boxed{\frac{98}{3} \text{ cubic feet}}$