Strengthened Analysis of Linear Sampling Method for Biharmonic Wave Scattering

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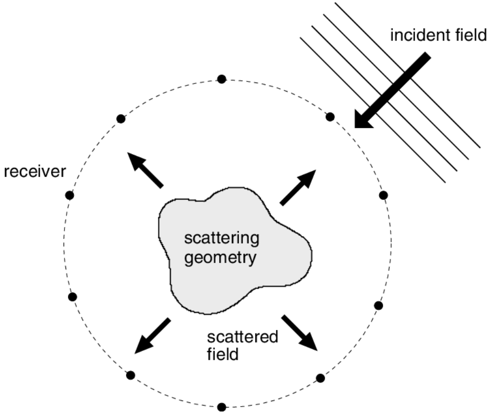
Overview

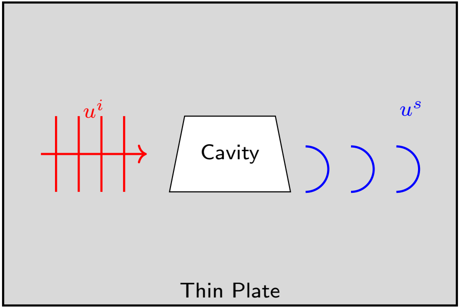
Inverse Problems for Waves

- Send waves to an object with unknown properties.
- Measure scattered waves.
- **Task:** Deduce information about the object (that's the inverse problem)



Wave Interaction with Thin Elastic Plates

- Thin elastic plate: Kirchhoff-Love theory pure-bending case, thickness H small compared with the dimensions defining the span of the surface in (x, y) (time-harmonic biharmonic wave equation)
- Cavity (unknown shape) perturbs vibrations
- **Task:** Reconstruct cavity shape from scattered plate waves



Selected Application: Structural Health Monitoring

⇝ Structural Health monitoring (SHM) systems use sensors and data analysis to assess structures, detect damage, and enhance safety

- early detection of delaminations & cracks
- safety enhancement
- reduction of maintenance intervals
- cost saving by optimal assembly of sensors & actuators

Sketch of experimental setup for an SHM system

Biharmonic Clamped Scattering Problem

- Scattering of time-harmonic flexural waves by a 2D cavity in a thin plate D with boundary $\Gamma = \partial D$:

$$\Delta^2 u(x) - \kappa^4 u(x) = 0, \quad x \in \mathbb{R}^2 \setminus \overline{D},$$
$$u(x) = 0, \quad \text{and} \quad \partial_\nu u(x) = 0,\ x \in \Gamma$$
$$\lim_{|x|\to\infty} \sqrt{|x|}\left(\frac{\partial u^s(x)}{\partial |x|} - i\kappa u^s(x)\right) = 0.$$

- κ :wavenumber ($\sim$ frequency)
- $u = u^i + u^s$:total wave, u^i :incident wave, u^s :scattered wave
- Incident plane wave: $u^i(x) = e^{i\kappa x\cdot d}$, $d \in \mathbb{S}^1$:incident direction.
- Clamped boundary conditions given by Cauchy data (Dirichlet problem)– keeps edges of the vibrating plate fixed.

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- Clamped boundary conditions given by Cauchy data (Dirichlet problem)– keeps edges of the vibrating plate fixed.
- Well-posed: Bourgeouis-Hazard 2019, Dong-Li 2024.

Far field measurements & inverse problem

- Far field asymptotic expansion.

$$u^s(x) = \frac{e^{i\kappa|x|}}{\sqrt{x}} u^\infty(\hat{x}) + O(|x|^{-3/2}), \quad |x| \to \infty, \hat{x} := \frac{x}{|x|}$$

$u^\infty(\hat{x}) = u^\infty(\hat{x}, d)$:the far field pattern of u^s (measured data)

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- The inverse problem: reconstruct D from measured far field patterns $u^\infty(\hat{x}, d, \kappa)$ for all $\hat{x}, d \in \mathbb{S}^1$ at a fixed frequency κ.

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- On the uniqueness: Dong-Li 2024.

Biharmonic Wave Decomposition

The biharmonic scattered field u^s can be split into two simpler pieces:

$$u^s = u^s_{\mathsf{H}} + u^s_{\mathsf{M}},$$

where

- u^s_{H}: the **Helmholtz component**, satisfies

$$\Delta u^s_{\mathsf{H}} + \kappa^2 u^s_{\mathsf{H}} = 0, \quad \mathbb{R}^2 \setminus \overline{D}$$

- u^s_{M}: the **modified Helmholtz component**, satisfies

$$\Delta u^s_{\mathsf{M}} - \kappa^2 u^s_{\mathsf{M}} = 0, \quad \mathbb{R}^2 \setminus \overline{D}$$

Key idea: Instead of tackling a single fourth-order PDE, we reduce the problem to two coupled *second-order* equations by factoring $\Delta^2 - \kappa^4 = (\Delta - \kappa^2)(\Delta - \kappa^2)$.

Coupled Scattering System

The biharmonic scattering problem can be reformulated as a coupled system of radiating solutions $(u^s_{\mathsf{H}}, u^s_{\mathsf{M}})$:

$$\begin{cases} \Delta u^s_{\mathsf{H}} + \kappa^2 u^s_{\mathsf{H}} = 0, \\ \Delta u^s_{\mathsf{M}} - \kappa^2 u^s_{\mathsf{M}} = 0, \end{cases} \qquad \text{in } \mathbb{R}^2 \setminus \overline{D},$$

with boundary conditions on ∂D:

$$u^s_{\mathsf{H}} + u^s_{\mathsf{M}} = -u^i, \quad \partial_\nu u^s_{\mathsf{H}} + \partial_\nu u^s_{\mathsf{M}} = -\partial_\nu u^i.$$

Key idea: u^s_{M} decays exponentially at infinity, so only u^s_{H} contributes to the far field. This reduction to a coupled second-order system greatly simplifies analysis and computation.

$$|u^s_{\mathsf{H}}| = O(r^{-1/2}) \quad \text{and} \quad |u_{\mathsf{M}}|^s = O(e^{-\kappa r})$$

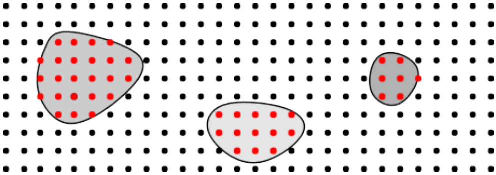
as $r = |x| \to \infty$. Moreover,

$$u^\infty = u^\infty_{\mathsf{H}}$$

Sampling Methods

Examples of sampling methods. *Linear Sampling Method* (Colton-Kirsch, 1996), *Factorization Method* (Kirsch 1998), *Probe Method* (Potthast, 2001), *Reciprocity Gap Method* (Colton-Haddar, 2005),...)

Principle: the idea is to construct an indicator test function $\mathcal{I}(z)$ that will test whether a sampling point z is in the interior or exterior of the scatterer (i.e. $\mathcal{I}(z) \approx 1$ inside scatterer, $\mathcal{I}(z) \approx 0$ outside scatterer).



(+) Non-iterative, the computation of $\mathcal{I}$ does not require a forward solver.
(-) Requires a large amount of multi-static data (many transmitters-receivers).

Prior Work on Sampling Methods for Biharmonic Scattering

- **LSM: Near-field approach** (Bourgeois, B. Chapuis & A. Recoquillay, (2020))
  - Uses *near-field measurements*: boundary measurements with point sources and dipoles.
  - they recovered cavities in a clamped and free plate
  - Limitation: Requires more dense measurement data.
- **LSM: Far-field approach** (Guo, J., Long, Y., Wu, Q., Li, J., (2024)
  - Uses *far-field measurements*
  - their analysis required excluding Dirichlet eigenvalues of the negative Laplacian & auxiliary eigenvalue problem.
- **Direct Sampling Method (DSM) with Far-Field Data** (Harris, I., Lee, H., Li, P.(2025))
  - recovered clamped data; their method more robust towards noise
  - their method enables resolution analysis of for reconstructions
- **Methods based on reverse time migration (RTM)** (Zhu, T. & Ge, Z. (2025))
  - applied to four types of boundary conditions on the obstacle
  - incorporated phased and phaseless measurement data (near-field/far-field)

Our contribution: - Strengthened LSM analysis with far-field measurements that relaxes previous assumptions. - Applies LSM framework without excluding Dirichlet eigenvalues. - Provides a more general reconstruction methodology for the biharmonic case. - Our contribution also modifies method to incorporate partial data (not in this talk!)

Linear Sampling Method (LSM) — Core Idea

Goal: Determine whether a sampling point $z \in \mathbb{R}^2$ lies inside the unknown cavity $D \subset \mathbb{R}^2$ by solving a system of ill-posed linear integral equations.

LSM Equation:

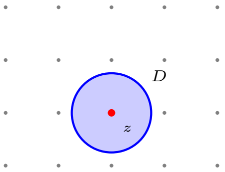
$$\mathcal{F} g_z = \gamma e^{-i\kappa \hat{x}\cdot z}, \quad (\mathcal{F} g_z)(\hat{x}) := \int_{\mathbb{S}^1} u^\infty(\hat{x}, d)\, g_z(d)\, ds(d), \quad g \in L^2(\mathbb{S}^1)$$

Where:

- $\mathcal{F}$: Far-field operator mapping weights g_z to superpositions of measured data.
- $\gamma e^{-i\kappa \hat{x}\cdot z}$: Far-field pattern of a point source at the sampling point z.
- $\gamma = -\frac{1}{2\kappa^2} \frac{e^{i\pi/4}}{\sqrt{8\pi\kappa}}$ is a constant arising from the fundamental solution

Sampling Principle:

- *Feasible* (regularized) solutions g_z exist with small norm if and only if $z \in D$.
- *Indicator:* Plotting $\|g_z\|_{L^2}$ reveals the support of D.



Ill-posedness and Approximate Solvability in LSM

Issue: The LSM equation

$$\mathcal{F}g_z = \gamma e^{-i\kappa\hat{x}\cdot z}$$

is ill-posed because the far-field operator $\mathcal{F}\colon L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is **compact**.

Why? Well-known that the The kernel $u^\infty(\hat{x}, d)$ of $\mathcal{F}$ is *analytic* in both variables $(\hat{x}, d) \in \mathbb{S}^1 \times \mathbb{S}^1$, leading to smoothing behavior and thus compactness.

Approximate Solvability Condition (Key Property)

For the LSM to function as a reliable indicator method, we want:

$$\mathcal{F} \text{ is injective with dense range in } L^2(\mathbb{S}^1).$$

This allows us to use regularized solutions g_z to test membership of $z \in D$.

Interpretation: While exact solutions g_z may not exist, we can seek *approximate* solutions whose norms reveal geometric information about the scatterer.

Injectivity via Clamped Transmission Problem

Goal: Show that if the far-field vanishes, the corresponding Herglotz kernel g must be zero ($\mathcal{F}g = 0 \Rightarrow g = 0$). Know by Rellich's lemma: $\mathcal{F}g = 0 \implies u^s_{\mathsf{H},g} = 0$ in $\mathbb{R}^2 \setminus \overline{D}$.

Idea: Decompose the scattered field $u^s_g = u^s_{\mathsf{H},g} + u^s_{\mathsf{M},g}$ of the farfield $\mathcal{F}g$:

- $u^s_{\mathsf{H},g}$ satisfies the Helmholtz equation $(\Delta + \kappa^2)u^s_{\mathsf{H},g} = 0$ in D and governs the far field.
- $u^s_{\mathsf{M},g}$ satisfies a modified Helmholtz equation $(\Delta - \kappa^2)u_{\mathsf{M}} = 0$ in $\mathbb{R}^2 \setminus \overline{D}$ and decays exponentially.

Clamped Transmission Problem:

$$(\Delta + \kappa^2)q = 0 \quad \text{in } D, \quad (\Delta - \kappa^2)p = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{D}$$

and

$$q + p = 0, \quad \partial_\nu q + \partial_\nu p = 0, \quad \text{on } \Gamma$$

with $q = u^s_{\mathsf{H},g}$ and $p = u^s_{\mathsf{M},g}$

Key Point

If the only solution is $q = p = 0$ (i.e., κ is **not** an eigenvalue of the clamped transmission problem), then $\mathcal{F}$ is injective. *In other words, the far field uniquely determines the kernel.*

Reciprocity and Dense Range of $\mathcal{F}$

Reciprocity Relation (Biharmonic Far-Field Pattern):

$$u^\infty(\hat{x}, d) = u^\infty(-d, -\hat{x})$$

(Follows from Green's representation theorem for far-field pattern and exponential decay of anti-Helmholtz component.)

Adjoint Relationship: The reciprocity identity implies:

$$(\mathcal{F}^*\varphi)(d) = \int_{\mathbb{S}^1} u^\infty(-d, -\hat{x})\, \varphi(\hat{x})\, ds(\hat{x}) = (\mathcal{F}\tilde{\varphi})(d), \quad \text{where } \tilde{\varphi}(\hat{y}) = \varphi(-\hat{y})$$

Conclusion: Know $\overline{\text{Range}(\mathcal{F})} = \text{Null}(\mathcal{F}^*)^\perp$, therefore:

Theorem (Injectivity and Density of $\mathcal{F}$)

Assume κ is not an eigenvalue of the clamped transmission problem. Then the far-field operator

$$\mathcal{F} : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$$

is injective and has dense range.

Outline of LSM

FarField Operator: $\mathcal{F} : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ defined by

$$\mathcal{F}g(\hat{x}) := \int_{\mathbb{S}^1} u^\infty(\hat{x}, d)\, g(d)\, ds(d).$$

Let us define for $(h_1, h_2)^\top \in H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$ the unique function $w \in H^2_{\text{loc}}(\mathbb{R}^2 \setminus \overline{D})$ satisfying

$$\begin{cases} \Delta^2 w - \kappa^4 w = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{D}, \\ w = h_1, \quad \partial_\nu w = h_2 \quad \text{on } \Gamma, \\ \lim\limits_{r \to \infty} \int_{|x|=r} \left| \dfrac{\partial w}{\partial r} - i\kappa w \right|^2 ds = 0. \text{ (SRC)} \end{cases} \tag{1}$$

$\Rightarrow Fg = w^\infty$ for w solving (1) with $(h_1, h_2)^\top = (v_g, \partial_\nu v_g)^\top$ on Γ, where

$$v_g(x) := \int_{\mathbb{S}^1} u^i(x, d) g(d)\, ds(d), \quad g \in L^2(\mathbb{S}^1), \quad x \in \mathbb{R}^2.$$

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$$\mathcal{F}g(\hat{x}) := \int_{\mathbb{S}^1} u^\infty(\hat{x}, d)\, g(d)\, ds(d).$$

$\Rightarrow$ Considering the (compact) operator $\mathcal{H} : L^2(\mathbb{S}^1) \to H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$ defined by

$$Hg := (v_g, \partial_\nu v_g)^\top$$

and the (compact) operator $\mathcal{G} : H^{3/2}(\Gamma) \times H^{1/2}(\Gamma) \to L^2(\mathbb{S}^1)$ by

$$\mathcal{G}(h_1, h_2)^\top := w^\infty,$$

then clearly:

$$\mathcal{F} = -\mathcal{G} \circ \mathcal{H}$$

Theorem: The operator $\mathcal{G} : H^{3/2}(\Gamma) \times H^{1/2}(\Gamma) \to L^2(\mathbb{S}^1)$, the clamped cavity D satisfies: a point $z \in D$ if and only if

$$\gamma\, e^{-i\kappa \hat{x} \cdot z} \in \text{Range}(\mathcal{G}).$$

Main Theorem of LSM

Theorem: Assume κ is not an eigenvalue of the clamped transmission problem. Then the operator $\mathcal{F}$ is injective with dense range. Moreover, the following holds.

- If $z \in D$ and $\epsilon > 0$ given then there exists g_z^α such that $||\mathcal{F}g_z^\alpha - \gamma e^{-i\kappa\hat{x}\cdot z}||_{L^2(\mathbb{S}^1)} \leq \epsilon$.
- If $z \notin D$ then for all g_z^α such that $||\mathcal{F}g_z^\alpha - \gamma e^{-i\kappa\hat{x}\cdot z}||_{L^2(\mathbb{S}^1)} \leq \epsilon$,

$$\lim_{\alpha\to 0} ||g_z^\alpha|| = \infty.$$

$\Rightarrow$ Gives a "characterization" of D in terms of nearby solutions of

$$\mathcal{F}g_z \simeq \gamma e^{-i\kappa\hat{x}\cdot z}$$

Drawbacks: this is not constructive ...

- Theorem doesn't provide how to construct g_z^α. In practice we use a regularization scheme.
- Have sufficient but not necessary criteria for recovering D. How does $z \mapsto ||g_z||_{L^2(\mathbb{S}^1)}$ behave inside D? (In practice, it's finite).

Numerical Implementation

- A regularization is needed to solve the far-field equation, e.g., we used Tikhonov reg. with $\alpha = 10^{-6}$ in all reconstructions

$$(\alpha I + F_d^* F_d) g_z^\alpha = F_d^* \phi_z$$

- Dimension of discretized matrix is based on the number N of sources/receivers. We selected a 250-by-250 grid for each construction.

  Example: 30 sources/receivers yields a 30-by-30 matrix F_d

- $F_d = [u^\infty(\hat{x}_i, \hat{y}_j)]_{i,j=1}^d$. Discretize so that

$$\hat{x}_i = \hat{y}_j = (\cos\theta_i, \sin\theta_j)\,, \quad \theta_i = 2\pi(i-1)/d,\, i = 1, \ldots, d.$$

  We used Li and Dong's boundary integral equation method to approximate the discretized far-field operator F_d.

- Add noise to test the stability of the LSM

$$F_d^\delta = [F_{i,j}(1 + \delta E_{i,j})]_{i,j=1}^d\,, \quad ||E||_2 = 1.$$

  $E \in \mathbb{C}^{d\times d}$ is a matrix with random entries, $0 < \delta \ll 1$ relative noise level

LSM Algorithm

The LSM reformulates the problem of determining the shape of the cavity D as the calculation of the indicator function g_z^α. The overall computational steps are summarized below:

Algorithm 1 Linear Sampling Method (LSM)

1. Choose a cutoff parameter $\zeta > 0$, and select a mesh $\mathcal{M}$ of sampling points in a region Ω that contains the cavity D;
2. For each sampling point $z \in \mathcal{M}$, compute an approximate solution g_z^α to the far-field equation using Tikhonov regularization with Morozov discrepancy principle;
3. Classify z as inside D if $1/||g_z^\alpha||_{L^2(\mathbb{S}^1)} > \zeta$, else outside.

Selected Numerical Result: Recovering the Apple-Shaped Cavity

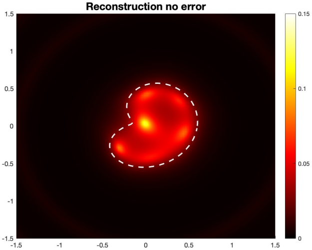

Figure: Recovering the Apple-Shaped Cavity with $\kappa = 2\pi$; no noise; 30 incident and observation directions; 250×250 grid

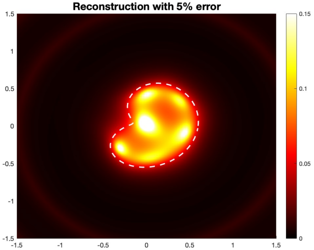

Figure: Recovering the Apple-Shaped Cavity with $\kappa = 2\pi$; noise $\delta = 0.05$; 30 incident and observation directions; 250×250 grid

Parametrization of Apple. $\gamma(t) = \frac{0.55(1+0.9\cos t+0.1\sin 2t}{1+0.75\cos t}(\cos t, \sin t)$

Numerical Result: Recovering the $\Gamma \in C^2$ Peach-Shaped Cavity

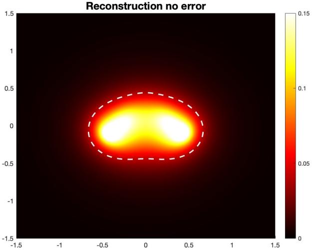

Figure: Recovering the Peach-Shaped Cavity with $\kappa = \pi$; no noise; 30 incident and observation directions; 250×250 grid

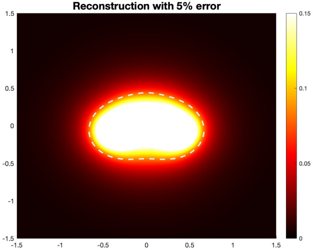

Figure: Recovering the Peach-Shaped Cavity with $\kappa = \pi$; noise $\delta = 0.05$; 30 incident and observation directions; 250×250 grid

Parametrization of Peach. $\gamma(t) = 0.22(\cos^2 t\sqrt{1-\sin t} + 2)(\cos t, \sin t)$

Numerical Result: Recovering the Unit Disk at Dirichlet eigenvalues

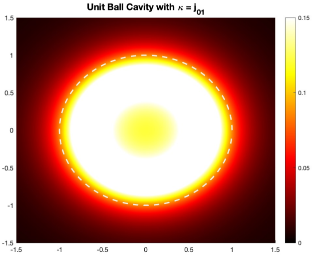

Figure: Reconstruction of the unit disk using the LSM. (a) $\kappa = j_{01}$

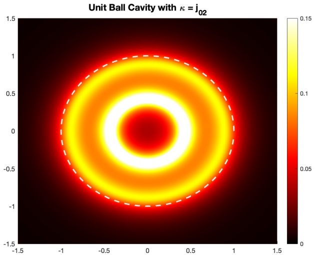

Figure: (b)Reconstruction of the unit disk using the LSM. $\kappa = j_{02}$

Consider the case where $\kappa^2 = \lambda$, with λ being an eigenvalue of the Dirichlet problem:

$$-\Delta\phi = \lambda\phi \quad \text{in } B_1(0), \quad \phi = 0 \quad \text{on } \partial B_1(0).$$

The eigenvalues of this problem are given by $\lambda_{mn} = j_{mn}^2$, where j_{mn} denotes the m-th positive zero of the Bessel function $J_n(r)$ of order n. j_{01} and j_{02} are the first and second positive zeros of the Bessel function $J_0(r)$

Conclusion & Future Work

Key Results:

- Applied **Linear Sampling Method (LSM)** to biharmonic scattering.
- Relaxed assumptions on the wavenumber κ compared to prior work.
- Modified framework in the same contribution for **partial data** (not covered in this talk).

Ongoing Work:

- **Factorization Method** (Kirsch) for rigorous necessary and sufficient conditions for recovery for **clamped obstacles**.
- Extension to **penetrable obstacles**: challenges involve coupled transmission conditions; goal is a fully rigorous LSM/FM framework for these cases.