A Factorization Method Approach to the Biharmonic Transmission Problem in Absorbing Media

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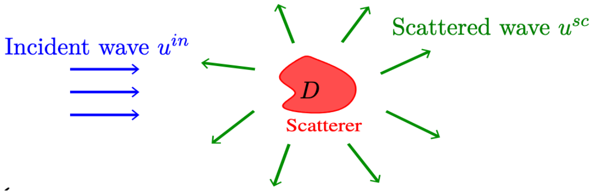
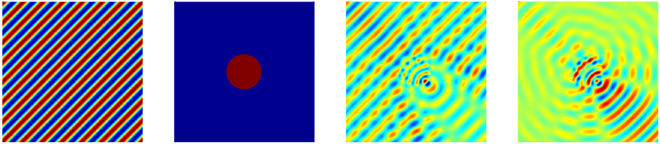
Paper on arXiv:

Direct & Inverse Scattering

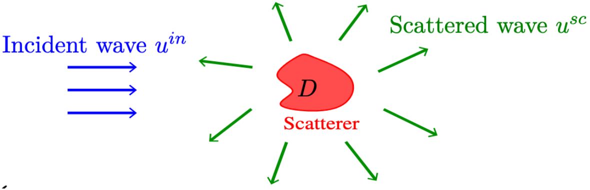


$$\begin{cases} u \text{ governed by a partial differential equation} \\ u = u^{\text{in}} + u^{\text{sc}} \quad \text{(total wave)} \\ u^{\text{sc}} \text{ is an outgoing wave} \quad \text{(radiation condition)} \end{cases}$$

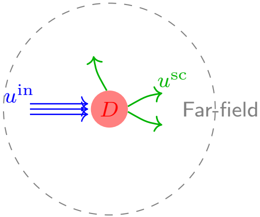


u^{in} scatterer u u^{sc}

Direct & Inverse Scattering

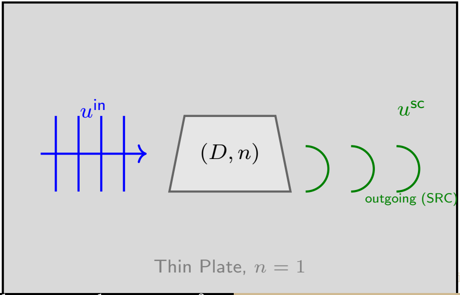


1. **Direct Problem:** Given incident wave u^{in} and scatterer D, find the scattered wave u^{sc}.
2. **Inverse Problem:** Given the (far-field) of scattered wave u^{sc}, find the scatterer D.



Setting: Biharmonic Transmission Problem

- Out-of-plane displacement $u = u^{\text{in}} + u^{\text{sc}}$ satisfies the **biharmonic transmission equation**
$$(\Delta^2 - \kappa^4 n(x))\, u(x) = 0 \quad \text{in } \mathbb{R}^2, \quad n \in L^\infty(\mathbb{R}^2),\ \operatorname{supp}(n-1) = \overline{D}.$$
$$u^{\text{in}}(x, d; \kappa) = \exp(i\kappa x \cdot d), \quad d \in \mathbb{S}^1, \quad \kappa > 0 : \text{wavenumber}$$
- **Transmission conditions on ∂D:**
$$[\![u]\!] = 0, \qquad [\![\partial_\nu u]\!] = 0,$$
$$[\![\Delta u]\!] = 0, \quad [\![\partial_\nu \Delta u]\!] = 0.$$
where $[\![\varphi]\!] = \varphi_+|_{\partial D} - \varphi_-|_{\partial D}$ denotes the jump across the interface.
- u^{sc} and Δu^{sc} satisfy a suitable **radiation condition** at infinity (SRC).



Absorbing Scatterer Condition and Well-Posedness

Direct Problem: Given the incident field u^i, find the scattered field u^s for a known penetrable medium (n, D):

$$(\Delta^2 - \kappa^4 n(x))\, u = 0 \text{ in } \mathbb{R}^2, \quad u = u^{\text{in}} + u^{\text{sc}}, \quad u^{\text{sc}} \,\&\, \Delta u^{\text{sc}} \text{ satisfies SRC.}$$

- Well-posedness (Fredholm of index zero) is established via a variational approach.
- **Uniqueness:** In the biharmonic case, requires the **absorbing scatterer condition**

$$\boxed{\Im(n(x)) \geq \alpha > 0 \quad \text{a.e. } x \in D}$$

which is *not* necessary in the acoustic case.

Theorem (Ceja-Ayala,R., Harris, I. & Sánchez-Vizuet, T., 2025)

For a penetrable medium (n, D) satisfying $\Im(n) \geq \alpha > 0$ a.e. in D and a given incident field u^{in}:

- The direct scattering problem has a unique solution u^{sc}.
- The solution operator is Fredholm of index zero.
- The scattered field u^{sc} satisfies the estimate

$$||u^{\text{sc}}||_{H^2(D)} \leq C||u^{\text{in}}||_{L^2(D)}$$

Lippmann–Schwinger Type Representation

Goal: Express the **unique** scattered field solution u^{sc} in terms of volume integral

Idea: Apply the Green's theorem for x in both the *interior* and *exterior* of the scatterer D. (Note χ_D is characteristic function over D)

Lippmann–Schwinger Type Representation

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Interior ($x \in D$)

$$u^{\mathsf{sc}}(x)\chi_D = \int_D G(x, z; \kappa)[\kappa^4 u^{\mathsf{sc}}(z) - \Delta^2 u^{\mathsf{sc}}(z)]\, \mathrm{d}z + \text{boundary terms on } \partial D.$$

Exterior ($x \notin D$)

$$u^{\mathsf{sc}}(x)(1 - \chi_D) = \text{boundary terms on } \partial D \cup \partial B_R.$$

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Combine both: adding the interior and exterior representations gives

Lippmann-Schwinger (L-S) Type Equation

$$\boxed{u^{\text{sc}}(x,d) = \kappa^4 \int_D (n(z)-1)\,G(x,z;\kappa)\,[u^{\text{in}}(z,d) + u^{\text{sc}}(z,d)]\,\mathrm{d}z.}$$

Interpretation:

- $G(x,z;\kappa)$ is the Green's function of biharmonic operator $(\Delta^2 - \kappa^4)$.

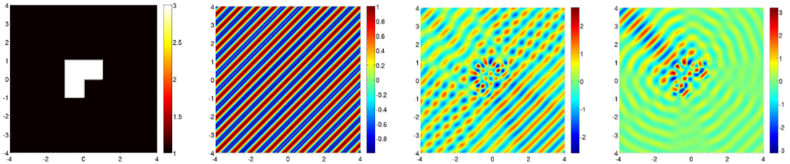
The Inverse Problem

Far-field intuition: Far from the scatterer, the scattered wave behaves like an outgoing cylindrical wave. Its amplitude and phase in each observation direction define the far-field pattern.

$$u^{\mathrm{sc}}(x,d) = \frac{e^{i\pi/4}}{\sqrt{8\pi\kappa}} \frac{e^{i\kappa|x|}}{\sqrt{|x|}} u^{\infty}(\hat{x},d;\kappa) + O(|x|^{-3/2}), \quad |x| \to \infty,\ \hat{x} = \frac{x}{|x|}.$$

- $u^{\infty}(\hat{x},d;\kappa)$: measured far-field pattern in direction of observation $\hat{x}$ for incident waves $d \in \mathbb{S}^1$
- Inverse problem: Recover the **'absorbing' penetrable scatterer** characterized by the pair $(n, D = \mathrm{supp}(n-1))$ — its shape, size, and location — given the full far-field data

$$\{u^{\infty}(\hat{x},d;\kappa) : \hat{x} \in \mathbb{S}^1,\ d \in \mathbb{S}^1\}.$$



Obstacle Incident wave Total wave Scattered wave

Adapted from: A. Lechleiter, *Making the Invisible Visible: Imaging Techniques for Inverse Problems* (2009)

We now define the (compact!) far-field operator as $F : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$

$$(Fg)(\hat{x}) = \int_{\mathbb{S}^1} u^\infty(\hat{x}, d; \kappa) g(d)\, ds(d).$$

Fg is far field corresponding to the incident field (Herglotz wave function)

$$v_g(x) = \int_{\mathbb{S}^1} e^{i\kappa x \cdot d} g(d)\, ds(d)$$

Note: v_g refers to superposition of plane waves.

Inverse Problem

The **inverse problem** also reads: given F find $D = \text{supp}(n-1)$!

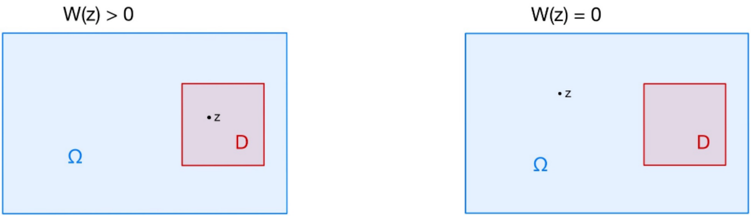
How to reconstruct D from F? Nonlinear optimization, local linearization, domain decomposition... **time-consuming!, Requires forward solver!**

Qualitative/Sampling Methods

Class of Reconstruction Methods

Qualitative Methods: Construct a binary criterium that decides whether a grid point z belongs to the support of the region in a computational simple manner.

- **Advantages:** Requires **little a priori** information on the scatterer. Very fast, no forward solver needed, fully direct method.
- **Disadvantages:** requires a lot of data, can only recover qualitative information.



The Range Test Property

Define the **Herglotz wave operator**

$$(Hg)(x) = v_g(x), \quad x \in D, \quad L^2(\mathbb{S}^1) \to L^2(D) \quad \text{bounded}$$

and its adjoint

$$(H^*\varphi)(\hat{x}) = \int_D e^{-i\kappa \hat{x}\cdot y}\,\varphi(y)\,dy, \quad \hat{x} \in \mathbb{S}^1, \quad L^2(D) \to L^2(\mathbb{S}^1) \quad \text{bounded}$$

For a sampling point $z \in \mathbb{R}^2$, consider the **range test equation**

$$H^*\varphi = \phi_z, \quad \text{where } \phi_z(\hat{x}) = e^{-i\kappa \hat{x}\cdot z}.$$

Note: ϕ_z is a **test function** and far-field pattern for the fundamental solution of the Helmholtz equation

Range Test Property

Let $D = \operatorname{supp}(n-1)$ be a bounded open set with connected complement. Then,

$$\boxed{\phi_z \in \operatorname{Range}(H^*) \iff z \in D.}$$

Key. Proof via a unique continuation argument.

H^* is a **bounded operator** depending on the (unknown) domain D.

Key Question

Can we relate $\operatorname{Range}(H^*)$, which depends on the unknown D, to the **known far-field operator** F?

Factorization Method Works by Factorization!

- Herglotz wave operator: $H : L^2(\mathbb{S}^1) \to L^2(D)$, $Hg = v_g|_D$
- Define $T : L^2(D) \to L^2(D)$ by

$$Tf = \frac{\kappa^2}{2}(n-1)(f + w(f)|_D)$$

where $(\Delta^2 - \kappa^4 n(x))w = \kappa^4(n-1)f$ in $\mathbb{R}^2$ + (SRCs)

Theorem

$$F = H^* T H$$

$$\begin{tikzcd}
L^2(\mathbb{S}^1) \arrow[r, "H"] \arrow[d, "F"'] & L^2(D) \arrow[d, "T"] \\
L^2(\mathbb{S}^1) & L^2(D) \arrow[l, "H^*"]
\end{tikzcd}$$

Task: Relate the Range(H^*) to the known operator F.

Factorization is From Lippmann–Schwinger

Starting point:

$$w(x) = \kappa^4 \int_D (n(y) - 1)\, G(x,y)\, [w(y) + f(y)]\, \mathrm{d}y.$$

Using the asymptotic form of $G(x,y)$ as $|x| \to \infty$:

$$G(x,y) \sim \frac{e^{i\kappa r}}{\sqrt{r}} \underbrace{\frac{\gamma}{2\kappa^2}\, e^{-i\kappa \hat{x}\cdot y}}_{\text{far–field term}}, \quad \gamma = \frac{e^{i\pi/4}}{\sqrt{8\pi\kappa}}$$

Then the far–field pattern becomes:

$$w^\infty(\hat{x}) = \frac{\kappa^2}{2} \int_D e^{-i\kappa\hat{x}\cdot y} (n-1)\, [f(y) + w(y)]\, \mathrm{d}y.$$

Hence, for a Herglotz incident field $f = Hg$,

$$(Fg)(\hat{x}) = \frac{\kappa^2}{2} \int_D e^{-i\kappa\hat{x}\cdot y} (n-1)\, [Hg(y) + w(y)]\, \mathrm{d}y.$$

Recognizing the operator structure:

$$F = H^*\, T\, H, \qquad Tf = \tfrac{\kappa^2}{2}(n-1)(f + w).$$

Key Property of the Middle Operator T

Operator factorization:

$$F = H^* \, T \, H, \qquad T f = \frac{\kappa^2}{2} \, (n-1) \, (f+w)$$

Define the imaginary parts:

$$\Im(F) := \frac{F - F^*}{2i}, \qquad \Im(T) := \frac{T - T^*}{2i},$$

which are self-adjoint & positive.

$$\Im(F) = H^* \, \Im(T) \, H$$

We have: Range$((\Im(F))^{1/2})$=Range(H^*), which follows from the following result.

Theorem ((Ceja-Ayala,R., Harris, I. & Ozochiawaeze, G. (2025))

Let the scatterer $D = \mathrm{supp}(n-1)$ *be* **absorbing**, *i.e.,*

$$\Im(n(x)) \geq \alpha > 0 \quad \text{a.e. } x \in D.$$

Then the middle operator $\Im(T)$ *is* **coercive** *on* $L^2(D)$*, that is,* $\exists \, \mu > 0$ *so that*

$$(\Im(T)f, f)_{L^2(D)} = \Im(Tf, f)_{L^2(D)} \geq \mu ||f||^2_{L^2(D)}$$

for all $f \in L^2(D)$.

Key. Proof by contradiction, exploit biharmonic wave decomposition for w obtained from factoring $(\Delta^2 - \kappa^4)w = (\Delta + \kappa^2)(\Delta - \kappa)w$

Reconstruction of D by the FM

We have the positive definite, compact operator

$$\Im(F) := \frac{F - F^*}{2i} : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1).$$

We also have

$$\text{Range}\big((\Im(F))^{1/2}\big) = \text{Range}(H^*)$$

so

$$z \in D \iff \phi_z \in \text{Range}\big((\Im(F))^{1/2}\big)$$

The Factorization Method (FM)

Let the pair $(\lambda_j, \psi_j) \in \mathbb{R}^+ \times L^2(\mathbb{S}^1)$ be the orthonormal eigensystem of $\Im(F)$. Then by the Picard range criterion we have:

$$z \in D \iff W(z) = \left[\sum_{j=1}^{\infty} \frac{\left| (\phi_z, \psi_j)_{L^2(\mathbb{S}^1)} \right|^2}{\lambda_j} \right]^{-1} > 0.$$

Thus, we have complete characterization of penetrable scatterer D:

$$\chi_D(z) = \text{sgn}(W(z)) := \begin{cases} 1, & x \in D \\ 0, & x \notin D \end{cases}$$

Born Approximation for Weak Scatterers

Integral operator:

$$(Kf)(x) := \int_D G(x,y)\,(n(y)-1)\,f(y)\,dy$$

$$\text{Assume } ||K|| < 1, \qquad (|D| < 1).$$

L-S Type Equation then becomes:

$$u^s = Ku^{\text{in}} + Ku^{\text{sc}} \iff u^s = (I-K)^{-1}Ku^{\text{in}}$$

Born approximation:

$$u^{\text{sc}} \approx \underbrace{Ku^{\text{in}}}_{\text{first-order}}$$

Far-field (linearized):

$$\begin{aligned} u^\infty(\hat{x},d) &\approx \frac{\kappa^2}{2}\int_D (n-1)\,e^{-i\kappa\hat{x}\cdot y}\,u^{\text{in}}(y)\,dy \\ &= \frac{\kappa^2}{2}\int_D (n-1)\,e^{-i\kappa\hat{x}\cdot y}\,e^{i\kappa d\cdot y}\,dy \\ &= \kappa^2/2\cdot\widehat{(n-1)}(\xi), \quad \xi := \kappa(d-\hat{x}) \end{aligned}$$

Key intuition: small/weak scatterer $\Rightarrow$ far-field $\approx$ Fourier transform of contrast.

Comparing Exact Far-Field & Born Approximation

Assume a small disk scatterer $D = B_\epsilon$, $0 < \epsilon \le 1$, with constant material parameter $n = 1.0 + 1.5i$. The direct scattering problem reads:

$$\Delta^2 u^{\mathsf{sc}} - \kappa^4 u^{\mathsf{sc}} = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{B_\epsilon}, \quad \Delta^2 u - \kappa^4 u = 0 \quad \text{in } B_\epsilon,$$

with boundary (transmission) conditions at $r = \epsilon$:

$$u^{\mathsf{sc}} - u = -u^{\mathsf{in}}, \quad \partial_r u^{\mathsf{sc}} - \partial_r u = -\partial_r u^{\mathsf{in}}, \quad \Delta u^{\mathsf{sc}} - \Delta u = -\Delta u^{\mathsf{in}}, \quad \partial_r \Delta u^{\mathsf{sc}} - \partial_r \Delta u = -\partial_r \Delta u^{\mathsf{in}}.$$

Expand incident plane wave via Jacobi–Anger:

$$u^{\mathsf{in}}(r,\theta) = \sum_{\ell=0}^{\infty} i^\ell J_\ell(\kappa r)\, e^{i\ell(\theta-\phi)}, \quad d = (\cos\phi, \sin\phi)$$

Separation of variables and the boundary conditions at $r = \epsilon$ give a linear system:

$$\mathbb{M}\mathbf{u} = \mathbf{f}, \quad \mathbf{u} = [a_\ell, b_\ell, c_\ell, d_\ell]^T, \quad \mathbf{f} = [-J_\ell(\kappa\epsilon), -\kappa J'_\ell(\kappa\epsilon), \kappa^2 J_\ell(\kappa\epsilon), \kappa^3 J'_\ell(\kappa\epsilon)]^T.$$

Once solved, the exact biharmonic far-field pattern is

$$u^\infty(\theta,\phi) = \frac{4}{\mathrm{i}} \sum_{\ell=0}^{\infty} a_\ell\, e^{i\ell(\theta-\phi)}.$$

Computing the Matrix $\mathbb{M}$ for a Disk Scatterer

Fourier Series Ansatz for the Scattered Field.
We decompose u^{sc} and u into propagative $((\Delta + \kappa^2)$ and evanescent $(\Delta - \kappa^2)$ components:

$$u_{\mathsf{H}}(r,\theta) = \sum_{|\ell|=0}^{\infty} i^{\ell} a_{\ell} H_{\ell}^{(1)}(\kappa r) e^{i\ell(\theta-\phi)}, \quad u_{\mathsf{M}}(r,\theta) = \sum_{|\ell|=0}^{\infty} i^{\ell} b_{\ell} H_{\ell}^{(1)}(i\kappa r) e^{i\ell(\theta-\phi)}.$$

Inside the Scatterer.
The total field $u = u_{\mathsf{pr}} + u_{\mathsf{ev}}$ is expanded as

$$u_{\mathsf{pr}}(r,\theta) = \sum_{|\ell|=0}^{\infty} i^{\ell} c_{\ell} J_{\ell}(\kappa n^{1/4} r) e^{i\ell(\theta-\phi)}, \quad u_{\mathsf{ev}}(r,\theta) = \sum_{|\ell|=0}^{\infty} i^{\ell} d_{\ell} J_{\ell}(i\kappa n^{1/4} r) e^{i\ell(\theta-\phi)}.$$

Key Observation.

$$\Delta u_{\mathsf{pr}} = -\kappa^2 \sqrt{n}\, u_{\mathsf{pr}}, \quad \Delta u_{\mathsf{ev}} = \kappa^2 \sqrt{n}\, u_{\mathsf{ev}}.$$

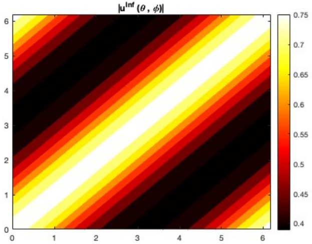
Linear System from Boundary Conditions.
At $r = \epsilon$, the four boundary conditions give

$$\mathbb{M}\mathbf{u} = \mathbf{f},$$

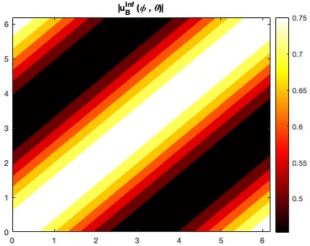
which determines the coefficients $a_{\ell}, b_{\ell}, c_{\ell}, d_{\ell}$.

Biharmonic Far-Field Pattern vs Born Approximation ($\varepsilon = 0.5$)

Exact Far-Field Pattern



Born Approximation



$$\mathbf{F}_{\text{exact}} = [u^{\infty}(\theta_i, \phi_j)]_{i,j=1}^{64}, \quad \mathbf{F}_{\text{Born}} = [u_{\text{B}}^{\infty}(\theta_i, \phi_j)]_{i,j=1}^{64},$$
$$\hat{x} = (\cos\theta, \sin\theta), \quad d = (\cos\phi, \sin\phi), \quad n = 1.0 + 1.5i$$

Absolute Error $\|\mathbf{F}_{\text{exact}} - \mathbf{F}_{\text{Born}}\|_{\infty}$ for Small Disks B_{ϵ}

ϵ	Error
1.00	0.6480
0.90	0.5008
0.80	0.3784
0.70	0.2774
0.60	0.1935
0.50	0.2134

FM Discretization and Imaging Functional

Algorithm 1 Factorization Method

Input: Wavenumber κ, will fix regularization $\alpha = 10^{-5}$, noise level δ.

Step 1: Sample Directions.

Compute 64 equally spaced angles $\theta_i = 2\pi(i-1)/64$, $\hat{x}_i = d_i = (\cos\theta_i, \sin\theta_i)$.

Step 2: Assemble Far-Field Matrix.

Evaluate $\mathbf{F}(i,j) = u^{\infty}(\hat{x}_i, d_j)$ and form $\mathbf{F} \in \mathbb{C}^{64\times 64}$.

Step 3: Add Noise (optional).

$\mathbf{F}^{\delta}(i,j) = \mathbf{F}(i,j)\big(1 + \delta\,\mathbf{R}(i,j)\big), \quad \mathbf{R} \in \mathbb{C}^{64\times 64}$(error matrix)

Step 4: Compute SVD of Imaginary Part.

$\Im(\mathbf{F}) = \frac{\mathbf{F}-\mathbf{F}^*}{2i} = \sum_j \sigma_j\, \mathbf{u}_j \mathbf{v}_j^*$.

Step 5: FM Indicator.

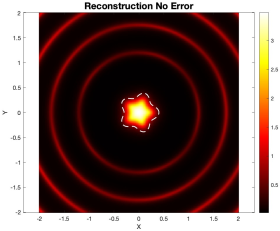
$$W_{\mathrm{FM}}(z) = \left[\sum_j \frac{\phi^2(\sigma_j;\alpha)}{\sigma_j} |(\mathbf{u}_j, \boldsymbol{\ell}_z)|^2\right]^{-1}, \quad \boldsymbol{\ell}_z = [e^{-i\kappa\hat{x}_i\cdot z}].$$

Tikhonov Filter Function

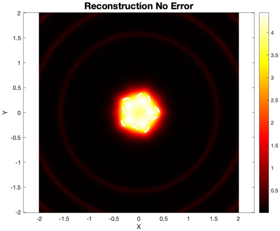
Tikhonov regularization: $\quad \phi(t;\alpha) = \dfrac{t^2}{t^2+\alpha}, \quad \alpha = 10^{-5}$

FM Reconstruction: Star-shaped Scatterer (Penetrable, $n = 2.5 + 1.5i$)

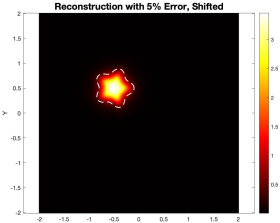
$k = 2\pi$ **(no noise)**



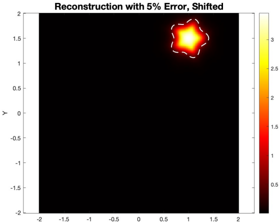
$k = 3\pi$ **(no noise)**



(5% noise, shifted (-0.5, 0.5))

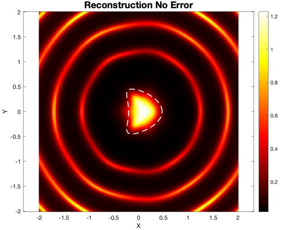


(5% noise, shifted (1, 1.5))

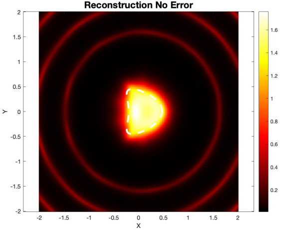


FM Reconstruction: Kite-shaped Scatterer (Penetrable, $n = 2.5 + 1.5i$)

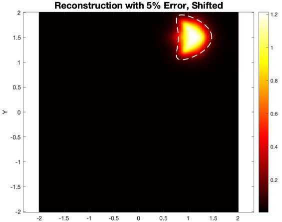
$k = 2\pi$ (no noise)



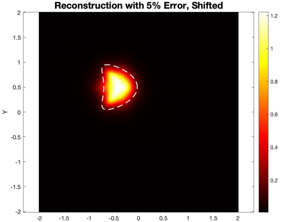
$k = 3\pi$ (no noise)



5% noise, shifted $(-0.5,\ 0.5)$

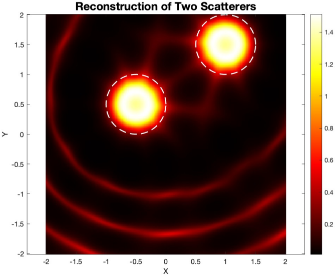


5% noise, shifted $(1,\ 1.5)$

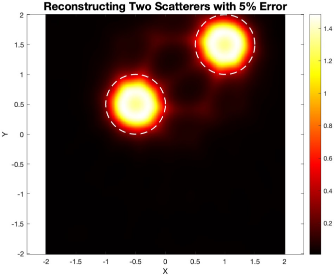


FM Reconstructions: Disk Scatterers (Penetrable, $\kappa = 2\pi$, $n = 2.5 + 1.5i$)

Reconstruction (No Noise)



Reconstruction (5% Noise)



FM reconstructions of a small disk scatterer $D = B_{1/2}$.

Conclusion

- Extended the **Factorization Method** (FM) to **biharmonic scattering** in **penetrable, absorbing media**.
- Provided **numerical reconstructions** showing FM's effectiveness for small scatterers and limited/noisy data.
- verified heuristically Born approximation valid first order linearization for weak scatterers.
- Open problems / future directions:
  - FM for **biharmonic scattering by impenetrable cavities** is still open.
  - Extending reconstruction methods to **near-field biharmonic scattering** for a penetrable medium remains open.
- Overall, FM provides a **qualitative reconstruction framework** beyond classical acoustic, elastic, & electromagnetic scattering, with extensions to complex media feasible.

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