Finite Element Modeling of Underwater Acoustic Environments

Based on my MS Thesis

General Ozochiawaeze

Department of Mathematics
Purdue University
Student Colloquium

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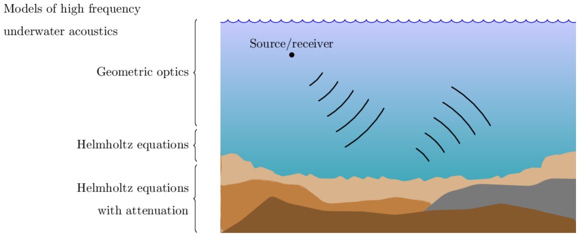
Overview

Forward Modeling

Forward problem: determine a model for acoustic propagation in an underwater environment consisting of a water layer and sediment layers.[1]



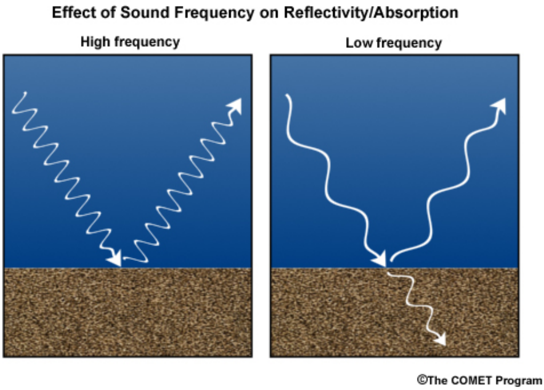
[1] Engquist et al., "Seafloor identification in sonar imagery via simulations of Helmholtz equations and discrete optimization".

Motivation

- This work incorporates finite element modeling of high frequency wave propagation in underwater acoustic environments.
- The forward modeling of underwater acoustic environments has important applications including:
  1. sonar imaging for target detection
  2. seafloor mapping
  3. sediment composition
  4. classification of marine habitats
  5. geoacoustic inversion
  6. noise prediction
- Want to simulate underwater acoustic wave scattering

High Frequency vs Low Frequency

High frequency sound reflects more efficiently than low frequency sound.



The Helmholtz Equation

Most problems of acoustic scattering can be formulated in the frequency domain as linear elliptic boundary value problems (BVPs).

The main character: **Helmholtz equation**

$$\Delta u + k^2 u = 0$$

u : acoustic pressure $\quad k := \frac{\omega}{c}$: wave number $\quad c$: sound speed
ω : frequency

Why Helmholtz Equation?

$$\Delta u + k^2 u = 0 \quad \text{in } \Omega \subset \mathbb{R}^2$$

Why is it interesting?

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1. Plenty of applications in computational ocean acoustics:
   - The Helmholtz equation is used, both analytically and numerically, to model underwater sound propagation.

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Why is it interesting?

1. Plenty of applications in computational ocean acoustics:
    - The Helmholtz equation is used, both analytically and numerically, to model underwater sound propagation.
2. difficult to solve numerically (for $k \gg 1$):
    - oscillating solutions $\Longrightarrow$ expensive to approximate
    - hard to formulate solvers for high-frequency solutions

Two Classes of Boundary Value Problems for the Helmholtz Equation

Two Classes of Boundary Value Problems for the Helmholtz Equation

1. **Interior Problems**
   - Helmholtz equation is set in a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, with Dirichlet, Neumann, or Impedance boundary conditions

Two Classes of Boundary Value Problems for the Helmholtz Equation

1. **Interior Problems**
    - Helmholtz equation is set in a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, with Dirichlet, Neumann, or Impedance boundary conditions
2. **Exterior Problems**
    - Helmholtz equation is set in an unbounded domain.
    - Most common and important example is the scattering problem.
    - Scattering problems concern propagation of waves that collide with some object.
    - Two types of acoustic scattering problems: sound-soft and sound-hard.

This work investigates both interior problems and exterior (scattering) problems, with greater emphasis on the latter.

Acoustic Scattering

Acoustic Scattering Boundary Value Problem

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \mathbb{R}^2 \setminus \overline{\Omega}, \\ \lim_{|\mathrm{x}|\to\infty} |\mathrm{x}| \left(\partial_{|\mathrm{x}|} u - iku \right) = 0 & \text{(SRC)} \\ u = -u_{inc} & \text{on } \partial\Omega \quad \text{(sound-soft)}, \\ \partial_{\mathrm{n}} u = -\partial_{\mathrm{n}} u_{inc} & \text{on } \partial\Omega \quad \text{(sound-hard)}. \end{cases}$$

k : wavenumber

Ω : the scatterer

$u_{inc}(\mathrm{x}) = -\exp(ik\hat{\theta} \cdot \mathrm{x})$: the plane incident wave

$\hat{\theta}$: incidence angle

$i = \sqrt{-1}$

SRC: Sommerfield Radiation Condition at infinity (outgoing scattered wave)

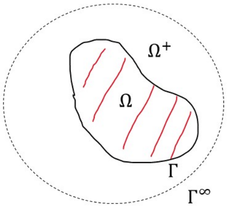
Absorbing Boundary Conditions

Approximation of Exterior (Scattering) Problems by Truncation

- for a numerical solution, need to truncate the unbounded domain and introduce an artificial surface with a boundary condition.
- An absorbing boundary condition (ABC) is an approximation to the Sommerfield radiation condition.

First Order ABC: $\frac{\partial u}{\partial n} - i\omega u = 0$ on Γ^∞

- Γ^∞: an artificial boundary.
- Ω^+: bounded domain enclosed by Γ^∞ and Γ.



Variational Formulations

- BVPs for linear elliptic PDEs like the Helmholtz equation are often posed in variational/weak form:

$$\text{(VF) find } u \in \mathcal{V} \text{ such that } a(u, v) = L(v) \quad \forall v \in \mathcal{V}.$$

- $\mathcal{V}$: Hilbert space
- $a(\cdot, \cdot) : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$: bilinear form
- $L : \mathcal{V} \to \mathbb{R}$: continuous linear functional

BVPs can be approximated using a weak Galerkin discretization:

$$\text{(WGD) find } u_N \in \mathcal{V}_N \text{ such that } a(u_N, v_N) = L(v_N) \quad \forall v_N \in \mathcal{V}_N.$$

- $\mathcal{V}_N \subset \mathcal{V}$: finite dimensional subspace, $\dim(\mathcal{V}_N) = N$

Well-Posedness of Variational Form

The weak problem is well-posed if: $\exists c_1, c_2 > 0$:

$$|a(u,v)| \leq c_1 \|u\|_{\mathcal{V}} \|v\|_{\mathcal{V}} \quad \forall u, v \in \mathcal{V}, \quad (continuity)$$
$$|a(v,v)| \geq c_2 \|v\|^2_{\mathcal{V}} \quad \forall v \in \mathcal{V}, \quad (coercivity)$$

By the Lax-Milgram Theorem,

- well-posedness of (VF): $\exists ! u \in \mathcal{V}, \|u\|_{\mathcal{V}} \leq \frac{\|L\|_{\mathcal{V}'}}{c_1}$;
- well-posedness of any (WGD): $\exists ! u_N \in \mathcal{V}_N, \|u_N\|_{\mathcal{V}} \leq \frac{\|L\|_{\mathcal{V}'}}{c_1}$,

where $\mathcal{V}'$ is the dual space of $\mathcal{V}$. Coercivity is a property of the bilinear form a.

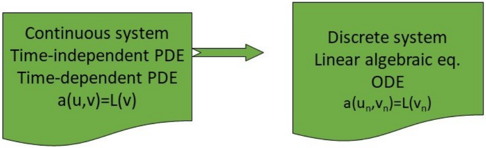
Overview of Finite Element Method

Finite element method (FEM) is a useful numerical method for modeling underwater acoustic scattering.

Main idea: take a geometrically complex domain Ω and divide it into cells and use polynomials (e.g., Lagrange or Hermite polynomials) for approximating a function over a cell.

Implementation:

1. FEnICS computing platform
2. COMSOL Multiphysics Acoustics Module



Overview of Finite Element Method

A finite element is defined as a triple $(\mathcal{K}, \mathcal{P}, \mathcal{N})$:

- $K \subseteq \mathbb{R}^n$: the element domain, a closed, bounded set with nonempty interior and a piecewise smooth boundary
- $\mathcal{P} \subset C(K)$: the shape functions, a finite-dimensional space of continuous functions on K
- $\mathcal{N}$: set of nodal variables, an indexed family of linear functionals on $\mathcal{P}$

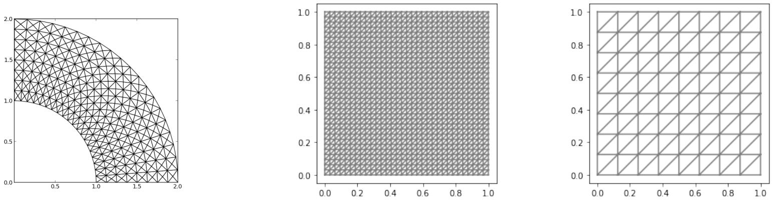


Figure: examples of piecewise linear $2D$ elements on a mesh

Numerical Results for Interior Problems

Consider the Dirichlet Helmholtz BVP

$$\begin{cases} \Delta u + k^2 u = -f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Weak formulation is

$$\begin{cases} a(u,v) := \int_\Omega \left(\nabla u \cdot \nabla v - k^2 uv \right) \, dx, \\ L(v) := \int_\Omega fv \, dx, \\ \mathcal{V} := H_0^1(\Omega) = \{ v \in H^1(\Omega) \, : \, v|_{\partial\Omega} = 0 \}. \end{cases}$$

How was the Helmholtz Weak Form Obtained?

(VF) was obtained by

1. multiplying $\mathcal{L}u := \Delta u + k^2 u = f$ with test function v;
2. using Green's 1st identity:

$$\int_\Omega [v\Delta u + \nabla v \cdot \nabla u] \, dx = \int_{\partial\Omega} v\nabla u \cdot \mathrm{n} \, dS$$

3. integrating by parts
4. substituting the boundary condition in the boundary term

Is the Helmholtz coercive?

- Continuity of $a(u, v)$ follows from the Cauchy-Schwarz inequality.

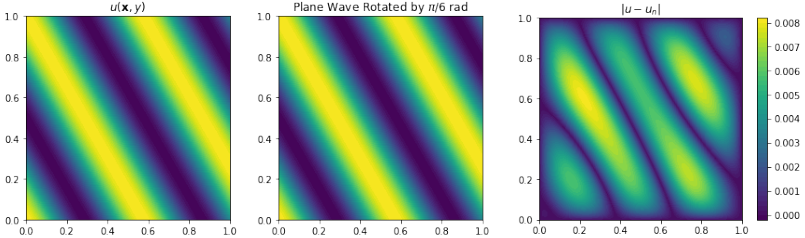
Is the Helmholtz coercive?

- Continuity of $a(u, v)$ follows from the Cauchy-Schwarz inequality.
- For $k^2 \geq \lambda_1 > 0$ (1st Laplace-Dirichlet eigenvalue), $a(\cdot, \cdot)$ is continuous but not coercive in $H_0^1(\Omega)$.
- Helmholtz equation is not coercive for large k. This makes high frequency problems hard!
- The Helmholtz problem is well-posed if k^2 is not an eigenvalue of $-\Delta$.

Numerical Results for Interior Problems

Dirichlet Problem:

FEM Solution   True Solution   Plot of Absolute Error



Numerical Results for Interior Problems

Consider also the Neumann Helmholtz BVP

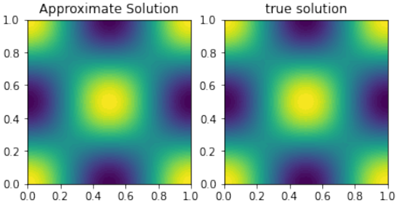
$$\begin{cases} \Delta u + k^2 u = -f & \text{in } \Omega, \\ \nabla u \cdot \mathrm{n} = 0 & \text{on } \partial\Omega. \end{cases}$$

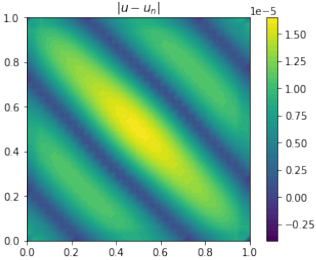
Weak formulation is

$$\begin{cases} a(u,v) := \int_\Omega \left(\nabla u \cdot \nabla v - k^2 uv \right) \, dx, \\ L(v) := \int_\Omega fv \; dx, \\ \mathcal{V} := H^1(\Omega) = \{ v \in L^2(\Omega) \, : \, \partial_{x_i} v \in L^2(\Omega), \, 1 \le i \le 2 \}. \end{cases}$$

Numerical Results for Interior Problems

Neumann Problem:





Numerical Results for Interior Problems

Table: Error at the Vertex Values of Mesh N with $P1$ elements

N	$\frac{\|u-u_n\|_{L^2}}{\|u\|_{L^2}}$	$\frac{\|u-u_n\|_{\infty}}{\|u_n\|_{\infty}}$
32	0.005409	0.00416
64	0.001358	0.00103
128	0.000339	0.000259
512	0.000021253	0.000001623
1024	5.313×10^{-6}	4.058×10^{-6}

Table: Error at the Vertex Values of Mesh N with $P2$ elements

N	$\frac{\|u-u_n\|_{L^2}}{\|u\|_{L^2}}$:	$\frac{\|u-u_n\|_{\infty}}{\|u_n\|_{\infty}}$:
32	6.93×10^{-5}	1.28×10^{-5}
64	8.62×10^{-6}	8.36×10^{-7}
128	1.07×10^{-6}	5.29×10^{-8}
512	1.68×10^{-8}	2.08×10^{-10}
1024	4.41×10^{-9}	3.73×10^{-11}

Point Source Problem: Helmholtz Equation With Attenuation Term

The next interior problem that serves as a model problem for generating circular waves is a Dirichlet Helmholtz BVP with a point source in the domain and zero boundary conditions on the square:

$$\begin{cases} \Delta u + (k^2 + ik\alpha)u = -\delta(\mathrm{x} - \mathrm{y}), \\ u = 0, \end{cases}$$

where $ik\alpha$ denotes the attenuation term and α is the attenuation/damping coefficient. The subsequent model produces waves that begin to "die out" before reaching the boundary.

Weak Formulation of Point Source Problem

For implementation, we split the solution $u = u_r + iu_i$, where $\Re u = u_r$, $\Im u = u_i$.

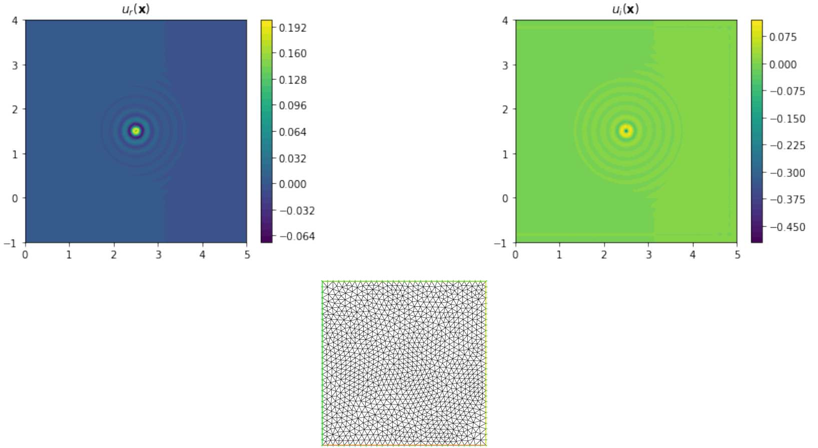
(VF) of Point Source Problem:

$$\begin{cases} a(u,v) := \int_\Omega (\nabla u_r \cdot \nabla v_r) + k^2 u_r v_r - \alpha k u_i v_r + (\nabla u_i \cdot \nabla v_i) + k^2 u_i v_i + \\ \alpha k u_r v_i \, d\mathbf{x}, \\ L(u,v) := \int_\Omega f v_r \, d\mathbf{x}, \\ \mathcal{V} := H_0^1(\Omega) \times H_0^1(\Omega) = \{(v_r, v_i) \in H^1(\Omega) \, : \, v_r|_{\partial\Omega} = v_i|_{\partial\Omega} = 0\}, \end{cases}$$

with $u = u_r + iu_i$ and $v = v_r + iv_i \in \mathbb{C}$. The standard Lagrange family of elements is used via a continuous Galerkin method.

Plot of Solution to Point Source Problem

Figure: Left: $\Re(u)$; Middle: Mesh $(N = 100)$; Right: $\Im(u)$

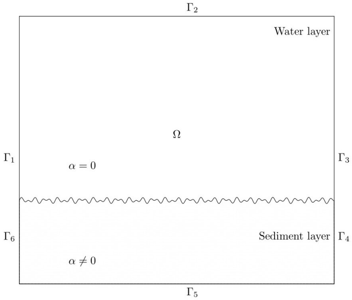


Damped Helmholtz equation $\Delta u + \omega^2 u + i\omega\alpha u = 0$

$$u = 0 \text{ on } \Gamma_5 \quad \text{(Dirichlet)}$$

$$\frac{\partial u}{\partial \mathrm{n}} - i\omega u = i\omega(1 - \sin\theta) u_{inc} \text{ on } \Gamma_2 \quad \text{(ABC)}$$

$$u\Big|_{\Gamma_1} = u\Big|_{\Gamma_3}, \quad u\Big|_{\Gamma_6} = u\Big|_{\Gamma_4}$$



- $u = u_{scat} + u_{inc}$
- u_{scat}: the scattered wave solution
- $u_{inc} = \exp(i\omega(\cos\theta x + \sin\theta y))$: the incoming plane wave

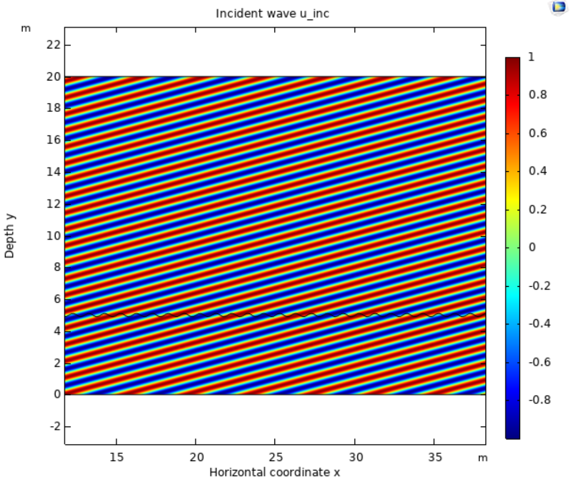
Weak Form of Water-Sediment Model

The weak formulation of the water-sediment model is

$$\begin{cases} a(u, v) := \displaystyle\int_\Omega \left(-\nabla u \cdot \overline{\nabla v} + \omega^2 u \overline{v} + i\omega\alpha u\right) dx, \\ L(u, v) := \displaystyle\int_{\Gamma_2} h\overline{v}\, ds, \\ \mathcal{V} := H^1(\Omega), \end{cases}$$

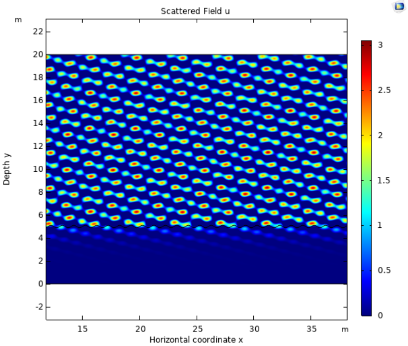
where $h = \nabla u_{inc} \cdot \mathrm{n}$.

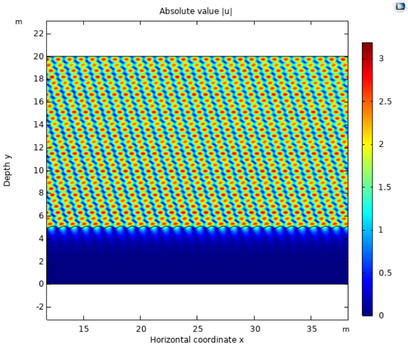
Plot of Incident Plane Wave



Simulation of Water-Sediment Domain

Sound attenuation occurs in the sediment.





Why Domain Decomposition?

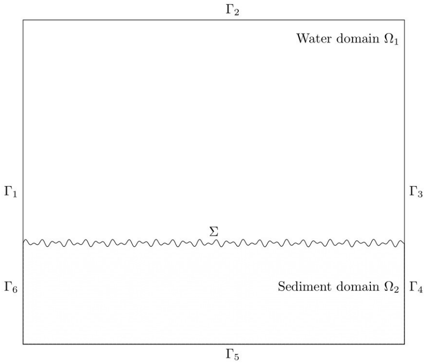
Main difficulties

- Helmholtz equation: highly-indefinite (not coercive!) at high wave numbers
- FEM leads to large, complex-valued, and highly indefinite sparse matrix for high wave numbers
- At high wave numbers/frequencies: solution will show non-robust behavior (pollution effect). Solutions become highly oscillatory at higher wave numbers

$\Longrightarrow$ **Alternative:** Domain Decomposition Methods!

Lions-Desprès DDM

Lions-Desprès DDM: a non-overlapping domain decomposition method used to solve high frequency acoustic scattering problems (Boubendir et al.). We split the domain of the water-sediment model into two non-overlapping subdomains $\Omega = \Omega_1 \cup \Omega_2$.



Lions-Deprès DDM Applied to the Water-Sediment Model

$$\begin{cases} \Delta u_1 + \omega^2 u_1 = 0 \quad (\text{on } \Omega_1) \\ \Delta u_2 + (\omega^2 + ia\omega) u_2 = 0 \quad (\text{on } \Omega_2) \\ \text{for the nth-iteration: } u_1, u_2 \text{ solve (SP) with the following condition on } \Sigma: \\ \partial_{n_1} u_1^n - i\omega u_1 = g_{12}^n \quad \text{on } \Sigma \\ \partial_{n_2} u_2^n - i\omega u_2 = g_{21}^n \quad \text{on } \Sigma \end{cases}$$

Weak Formulation of Water Subdomain

At the nth-iteration, we multiplied by a test function $v \in \mathcal{V}$ and found $u_1 \in \mathcal{V}$ such that

$$\begin{cases} \mathcal{V} = \left\{ v \in H^1(\Omega) \ : \ v\big|_{\Gamma_1} = v\big|_{\Gamma_3} \right\}, \\ a(u_1, v) = \displaystyle\int_{\Omega_1} \nabla u_1 \cdot \nabla v - \omega^2 u_1 v \; dx - i\omega \int_{\Sigma} u_1 v \; dS, \\ L(v) = \displaystyle\int_{\Gamma_2} hv \, dS + \int_{\Sigma} g_{12} v \, dS, \end{cases}$$

where $h = \nabla u_{inc} \cdot \mathrm{n}$.

Weak Formulation of the Sediment Subdomain

At the nth-iteration, we multiplied by a test function $v \in \mathcal{W}$ and found $u_2 \in \mathcal{W}$ such that

$$\begin{cases} \mathcal{W} = \left\{ v \in H^1(\Omega) \ : \ v\big|_{\Gamma_6} = v\big|_{\Gamma_4} \text{ and } v\big|_{\Gamma_5} = 0 \right\}, \\ a(u_2, v) = \int_{\Omega_2} \nabla u_2 \cdot \nabla v - (\omega^2 + i\omega\alpha) u_2 v \, dx - i\omega \int_{\Sigma} u_2 v \, dS \\ L(v) = \int_{\Sigma} g_{21} v \, dS. \end{cases}$$

Numerical Result with Lions-Despres

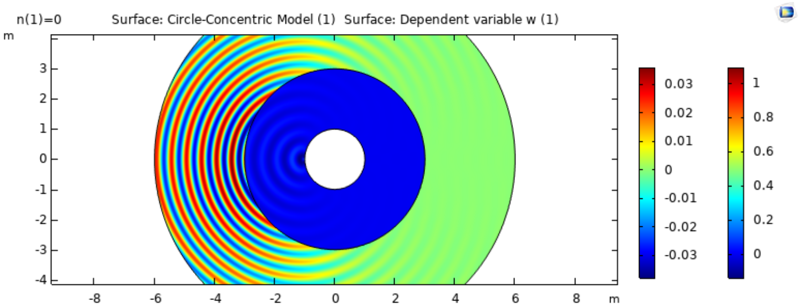


Figure: A 2*d* circle-concentric model of a plane wave by a unit sound-soft circular cylinder.

Conclusion and Future Work

Conclusion

- Forward modeling of both interior and exterior BVPs for the Helmholtz equation, simulating plane and circular waves arising in sonar imaging and noise detection
- Simulated a two-layer seafloor with attenuation in the sediment
- Formulation of the forward problem with Lions-Deprès DDM, which paves the way for detailed simulations of seafloor scattering with high frequencies

Conclusion and Future Work

Future Work

- implementation
- investigate the impact of the domain decomposition method on computational efficiency
- what is the number of iterations necessary to improve detailed simulation of underwater acoustic wave scattering?

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Thanks for Listening!